

A topological set theory implied by ZF and GPK_∞^+

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Abstract

We present a system of axioms motivated by a topological intuition: The set of subsets of any set is a topology on that set. On the one hand, this system is a common weakening of Zermelo-Fraenkel set theory ZF, the positive set theory GPK_∞^+ and the theory of hyperuniverses. On the other hand, it retains most of the expressiveness of these theories and has the same consistency strength as ZF. We single out the additional axiom of the universal set as the one that increases the consistency strength to that of GPK_∞^+ and explore several other axioms and interrelations between those theories. Our results are independent of whether the empty class is a set and whether atoms exist.

This article is a revised version of the first part of the author's doctoral thesis [Fac12].

Introduction

An axiomatic set theory can be thought of as an effort to make precise which classes are sets. It simultaneously aims at providing enough freedom of construction for all of classical mathematics and still remain consistent. It therefore must imply that all “reasonable” class comprehensions $\{x \mid \phi(x)\}$ produce sets and explain why the Russell class $\{x \mid x \notin x\}$ does not. The answer given by e. g. Zermelo-Fraenkel set theory (ZF) is the *Limitation of Size Principle*: Only small classes are sets. However, the totality of all mathematical objects, the *universe* $\mathbb{V} = \{x \mid x=x\}$, is a proper class in ZF.

Two very different ideas of sethood lead to a family of theories which do allow $\mathbb{V} \in \mathbb{V}$:

Firstly, one might blame the negation in the formula $x \notin x$ for Russell's paradox. The collection of *generalized positive formulas* is recursively defined by several construction steps not including negation. If the existence of $\{x \mid \phi(x)\}$ is stipulated for every generalized positive formula, a beautiful “positive” set theory emerges.

Secondly, instead of demanding that every class is a set, one might settle for the ability to approximate it by a least superset, a *closure* in a topological sense.

Surprisingly such “topological” set theories tend to prove the comprehension principle for generalized positive formulas, and conversely, in positive set theory, the universe is a topological space. More precisely, the sets are closed with respect to intersections and finite unions, and the universe is a set itself, so the sets represent the closed subclasses of a topology on \mathbb{V} . A class is a set if and only if it is topologically closed.

The first model of such a theory was constructed by R. J. Malitz in [Mal76] under the condition of the existence of certain large cardinal numbers. E. Weydert, M. Forti and R. Hinnion were able to show in [Wey89, FH89] that in fact a weakly compact cardinal suffices. In [Ess97] and [Ess99], O. Esser exhaustively answered the question of consistency for a specific positive set theory, GPK_∞^+ with a choice principle, and showed that it is mutually interpretable with a variant of Kelley-Morse set theory.

Atoms, sets, classes and topology

We incorporate proper classes into all the theories we consider. This enables us to write down many arguments in a more concise yet formally correct way, and helps separate the peculiarities of particular theories from the common assumptions about atoms, sets and classes.

We use the language of set theory with atoms, whose non-logical symbols are the binary relation symbol \in and the constant symbol \mathbb{A} . We say “ X is an element of Y ” for $X \in Y$. We call X an *atom* if $X \in \mathbb{A}$, and otherwise we call X a *class*. If a class is an element of any other class, it is a *set*; otherwise it is a *proper class*.

We denote the objects of our theories – all atoms, sets and classes – by capital letters and use lowercase letters for sets and atoms only, so:

- $\forall x \phi(x)$ means $\forall X. (\exists Y X \in Y) \Rightarrow \phi(X)$ and
- $\exists x \phi(x)$ means $\exists X. (\exists Y X \in Y) \wedge \phi(X)$.

For each formula ϕ let ϕ^C be its relativization to the sets and atoms, that is, every quantified variable in ϕ is replaced by a lowercase variable in ϕ^C .

Free variables in formulas that are supposed to be sentences are implicitly universally quantified. For example, we usually omit the outer universal quantifiers in axioms.

Using these definitions and conventions, we can now state the basic axioms concerning atoms, sets and classes. Firstly, we assume that classes are uniquely defined by their extension, that is, two classes are equal iff they have the same elements. Secondly, atoms do not have any elements. Thirdly, there are at least two distinct sets or atoms. And finally, any collection of sets and atoms which can be defined in terms of sets, atoms and finitely many fixed parameters, is a class. Formally:

<i>Extensionality</i>	$(X, Y \notin \mathbb{A} \wedge \forall Z. Z \in X \Leftrightarrow Z \in Y) \Rightarrow X = Y$
<i>Atoms</i>	$X \in \mathbb{A} \Rightarrow Y \notin X$
<i>Nontriviality</i>	$\exists x, y \ x \neq y$
<i>Comprehension(ψ)</i>	$\exists Z \notin \mathbb{A}. \forall x. x \in Z \Leftrightarrow \psi(x, \vec{P})$ for all formulas $\psi = \phi^C$

We will refer to these axioms as the *class axioms* from now on. Note that the object \mathbb{A} may well be a proper class, or a set. The atoms axiom implies however that \mathbb{A} is not an atom.

We call the axiom scheme given in the fourth line the *weak comprehension* scheme. It can be strengthened by removing the restriction on the formula ψ , instead allowing ψ to be any formula – even quantifying over all classes. Let us call that variant the *strong comprehension* scheme. The axiom of extensionality implies the uniqueness of the class Z . We also write $\{w \mid \psi(w, \vec{P})\}$ for Z , and generally use the customary notation for comprehensions, e.g. $\{x_1, \dots, x_n\} = \{y \mid y = x_1 \vee \dots \vee y = x_n\}$ for the class with finitely many elements x_1, \dots, x_n , $\emptyset = \{w \mid w \neq w\}$ for the empty class and $\mathbb{V} = \{w \mid w = w\}$ for the universal class. Also let $\mathbb{T} = \{x \mid \exists y. y \in x\}$ be the class of nonempty sets. The weak comprehension scheme allows us to define unions, intersections and differences in the usual way.

Given the class axioms, we can now define several topological terms. They all make sense in this weak theory, but one has to carefully avoid for now the assumption that any class is a set. Also, the “right” definition of a topology in our context is the collection of all nonempty closed sets instead of all open sets.

For given classes A and T , we call A *T-closed* if $A = \emptyset$ or $A \in T$. A *topology* on a class X is a class T of nonempty subsets of X , such that:

- X is T -closed.
- $\bigcap B$ is T -closed for every nonempty class $B \subseteq T$.
- $a \cup b$ is T -closed for all T -closed sets a and b .

The class X , together with T , is then called a *topological space*. If A is a T -closed class, then its complement $\complement A = X \setminus A$ is *T-open*. A class which is both T -closed and T -open is *T-clopen*.

The intersection of all T -closed supersets of a class $A \subseteq X$ is the least T -closed superset and is called the T -closure $\text{cl}_T(A)$ of A . Then $\text{int}_T(A) = \text{Ccl}_T(\text{C}A)$ is the largest T -open subclass of A and is called the T -interior of A . Every A with $x \in \text{int}_T(A)$ is a T -neighborhood of the point x . The explicit reference to T is often omitted and X itself is considered a topological space, if the topology is clear from the context.

If $S \subset T$ and both are topologies, we call S *coarser* and T *finer*. An intersection of several topologies on a set X always is a topology on X itself. Thus for every class B of subsets of X , if there is a coarsest topology $T \supseteq B$, then that is the intersection of all topologies S with $B \subseteq S$. We say that B is a *subbase* for T and that T is *generated* by B .

If $A \subseteq X$, we call a subclass $B \subseteq A$ *relatively closed* in A if there is a T -closed C such that $B = A \cap C$, and similarly for *relatively open* and *relatively clopen*. If every subclass of A is relatively closed in A , we say that A is *discrete*. Thus a T -closed set A is discrete iff all its nonempty subclasses are elements of T . Note that there is an equivalent definition of the discreteness of a class $A \subseteq X$ which can be expressed without quantifying over classes: A is discrete iff it contains none of its accumulation points, where an *accumulation point* is a point $x \in X$ which is an element of every T -closed $B \supseteq A \setminus \{x\}$. Formally, A is discrete iff it has at most one point or:

$$\forall x \in A \exists b \in T. A \subseteq b \cup \{x\} \wedge x \notin b$$

A topological space X is T_1 if for all distinct $x, y \in X$ there exists an open $U \subseteq X$ with $y \notin U \ni x$, or equivalently, if every singleton $\{x\} \subseteq X$ is closed. X is T_2 or *Hausdorff* if for all distinct $x, y \in X$ there exist disjoint open $U, V \subseteq X$ with $x \in U$ and $y \in V$. It is *regular* if for all closed $A \subseteq X$ and all $x \in X \setminus A$ there exist disjoint open $U, V \subseteq X$ with $A \subseteq U$ and $x \in V$. X is T_3 if it is regular and T_1 . It is *normal* if for all disjoint, closed $A, B \subseteq X$ there exist disjoint open $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$. X is T_4 if it is normal and T_1 .

A map $f : X \rightarrow Y$ between topological spaces is *continuous* if all preimages $f^{-1}[A]$ of closed sets $A \subseteq Y$ are closed, and it is *closed* if all images $f[A]$ of closed sets $A \subseteq X$ are closed.

Let \mathcal{K} be any class. We consider a class A to be \mathcal{K} -small if it is empty or there is a surjection from a member of \mathcal{K} onto A , that is:

$$A = \emptyset \quad \vee \quad \exists x \in \mathcal{K} \exists f : x \rightarrow A \quad f[x] = A$$

Otherwise, A is \mathcal{K} -large. We say \mathcal{K} -few for “a \mathcal{K} -small collection of”, and \mathcal{K} -many for “a \mathcal{K} -large collection of”. Although we quantified over classes in this definition, we will only use it in situations where there is an equivalent first-order formulation.

If all unions of \mathcal{K} -small subclasses of a topology T are T -closed, then T is called \mathcal{K} -additive or a \mathcal{K} -topology. If T is a subclass of every \mathcal{K} -topology $S \supseteq B$ on X , then T is \mathcal{K} -generated by B on X and B is a \mathcal{K} -subbase of T on X . If every element of T is an intersection of elements of B , B is a *base* of T .

A topology T on X is \mathcal{K} -compact if every T -cocover has a \mathcal{K} -small T -subcocover, where a T -cocover is a class $B \subseteq T$ with $\bigcap B = \emptyset$. Dually, we use the more familiar term *open cover* for a collection of T -open classes whose union is X , where applicable.

For all classes A and T , let

$$\square_T A = \{b \in T \mid b \subseteq A\} \quad \text{and} \quad \diamond_T A = \{b \in T \mid b \cap A \neq \emptyset\}.$$

If T is a topology on X , and if for all $a, b \in T$ the classes $\square_T a \cap \diamond_T b$ are sets, then the set $T = \square_T X = \diamond_T X$, together with the topology S \mathcal{K} -generated by $\{\square_T a \cap \diamond_T b \mid a, b \in T\}$ is called the \mathcal{K} -hyperspace (or *exponential space*) of X and denoted by $\text{Exp}_{\mathcal{K}}(X, T) = \langle \square_T X, S \rangle$, or in the short form: $\text{Exp}_{\mathcal{K}}(X)$. Since $\square_T a = \square_T a \cap \diamond_T X$ and $\diamond_T a = \square_T X \cap \diamond_T a$, the classes $\square_T a$ and $\diamond_T a$ are also sets and constitute another \mathcal{K} -subbase of the exponential \mathcal{K} -topology. A notable subspace of $\text{Exp}_{\mathcal{K}}(X)$ is the space $\text{Exp}_{\mathcal{K}}^c(X)$ of \mathcal{K} -compact subsets. In fact, this restriction suggests the canonical definition $\text{Exp}_{\mathcal{K}}^c(f)(a) = f[a]$ of a map $\text{Exp}_{\mathcal{K}}^c(f) : \text{Exp}_{\mathcal{K}}^c(X) \rightarrow \text{Exp}_{\mathcal{K}}^c(Y)$ for every continuous $f : X \rightarrow Y$, because continuous images of \mathcal{K} -compact sets are \mathcal{K} -compact. Moreover, $\text{Exp}_{\mathcal{K}}^c(f)$ is continuous itself.

\mathcal{K} should be pictured as a cardinal number, but prior to stating the axioms of essential set theory, the theory of ordinal and cardinal numbers is not available. However, to obtain useful ordinal

numbers, an axiom stating that the additivity is greater than the cardinality of any discrete set is needed. Fortunately, this can be expressed using the class \mathcal{D} of all discrete sets as the additivity.

1 Essential Set Theory

Consider the following, in addition to the class axioms:

<i>1st Topology Axiom</i>	$\mathbb{V} \in \mathbb{V}$
<i>2nd Topology Axiom</i>	If $A \subseteq \mathbb{T}$ is nonempty, then $\bigcap A$ is \mathbb{T} -closed.
<i>3rd Topology Axiom</i>	If a and b are \mathbb{T} -closed, then $a \cup b$ is \mathbb{T} -closed.
T_1	$\{a\}$ is \mathbb{T} -closed.
<i>Exponential</i>	$\Box_{\mathbb{T}} a \cap \Diamond_{\mathbb{T}} b$ is \mathbb{T} -closed.
<i>Discrete Additivity</i>	$\bigcup A$ is \mathbb{T} -closed for every \mathcal{D} -small class A .

We call this system of axioms *topological set theory*, or in short: \mathbb{TS} , and the theory \mathbb{TS} without the 1st topology axiom *essential set theory* or \mathbb{ES} . We will mostly work in \mathbb{ES} and explicitly single out the consequences of $\mathbb{V} \in \mathbb{V}$.

In \mathbb{ES} , the class $\mathbb{T} = \Box_{\mathbb{T}} \mathbb{V} = \Diamond_{\mathbb{T}} \mathbb{V}$ of all nonempty sets satisfies all the axioms of a topology on \mathbb{V} , except that it does not need to contain \mathbb{V} itself. Although it is not necessarily a class, we can therefore consider the collection of \mathbb{V} and all nonempty sets a topology on \mathbb{V} and informally attribute topological notions to it. We will call it the *universal topology* and whenever no other topology is explicitly mentioned, we will refer to it. Since no more than one element distinguishes the universal topology from \mathbb{T} , any topological statement about it can easily be reformulated as a statement about \mathbb{T} and hence be expressed in our theory. Having said this, we can interpret the third axiom as stating that the universe is a T_1 space.

Alternatively one can understand the axioms without referring to collections outside the theory's scope as follows: Every set a carries a topology $\Box a$, and a union of two sets is a set again. Then the T_1 axiom says that all sets are T_1 spaces (and that all singletons are sets) and the fourth says that every set's hyperspace exists.

If \mathbb{V} is not a set, we cannot interpret the exponential axiom as saying that the universe's hyperspace exists! Since $\Box a = \Box a \cap \Diamond a$, it implies the power set axiom, but it does not imply the sethood of $\Diamond a$ for every set a .

A very handy implication of the 2nd topology axiom and the exponential axiom is that for all sets b, c and every class A ,

$$\{x \in c \mid A \subseteq x \subseteq b\} = \begin{cases} c \cap \bigcap_{y \in A} (\Box b \cap \Diamond \{y\}) & \text{if } A \neq \emptyset. \\ (c \cap \Box b) \cup (c \cap A) & \text{if } A = \emptyset. \end{cases}$$

is closed, given that $c \cap A$ is closed or A is nonempty.

An important consequence of the T_1 axiom is that for each natural number¹ n , all classes with at most n elements are discrete sets. In particular, pairs are sets and we can define ordered pairs as Kuratowski pairs $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$. We adopt the convention that the $n+1$ -tuple $\langle x_1, \dots, x_{n+1} \rangle$ is $\langle \langle x_1, \dots, x_n \rangle, x_{n+1} \rangle$ and that relations and functions are classes of ordered pairs. With these definitions, all functional formulas ϕ^c on sets correspond to actual functions, although these might be proper classes. We denote the \in -relation for sets by $\mathbf{E} = \{\langle x, y \rangle \mid x \in y\}$, and the equality relation by $\Delta = \{\langle x, y \rangle \mid x = y\}$. Also, we write Δ_A for the equality $\Delta \cap A^2$ on a class A .

We have not yet made any stronger assumption than T_1 about the separation properties of sets. However, many desirable set-theoretic properties, particularly with respect to Cartesian products, apply only to Hausdorff sets, that is, sets whose natural topology is T_2 .

¹Until we have defined them in essential set theory, we consider natural numbers to be metamathematical objects.

We denote by $\square_{<n}A$ the class of all $b \subseteq A$ with less than n elements. Given $t_1, \dots, t_m \in \{1, \dots, n\}$. We define:

$$F_{n,t_1,\dots,t_m} : \mathbb{V}^n \rightarrow \mathbb{V}^m, F_{n,t_1,\dots,t_m}(x_1, \dots, x_n) = \langle x_{t_1}, \dots, x_{t_m} \rangle$$

With the corresponding choice of t_1, \dots, t_m , all projections and permutations can be expressed in this way.

For a set a , let a' be its *Cantor-Bendixson derivative*, the set of all its accumulation points, and let $a_I = a \setminus a'$ be the class of all its isolated points.

Proposition 1 (ES). *Let a and b be Hausdorff sets and $a_1, \dots, a_n \subseteq a$.*

1. $\square_{<n}a$ is a Hausdorff set.
2. The Cartesian product $a_1 \times \dots \times a_n$ is a Hausdorff set, too, and its universal topology is at least as fine as the product topology.
3. Every continuous function $F : a_1 \rightarrow a_2$ is a set.
4. For all $t_1, \dots, t_m \in \{1, \dots, n\}$, the function

$$F_{n,t_1,\dots,t_m} \upharpoonright a^n : a^n \rightarrow a^m$$

is a Hausdorff set. It is even closed with respect to the product topology of a^{n+m} .

5. For each $x \in a_I$, let $b_x \subseteq b$. Then for every map $F : a_I \rightarrow b$, the class $F \cup (a' \times b)$ is \mathbb{T} -closed and we can define the product as follows:

$$\prod_{x \in a_I} b_x = \{F \cup (a' \times b) \mid F : a_I \rightarrow b, \forall x F(x) \in b_x\}$$

It is \mathbb{T} -closed and its natural topology is at least as fine as its product topology.

Proof. (1): To show that it is a set it suffices to prove that it is a closed subset of the set $\square a$, so assume $b \in \square a \setminus \square_{<n}a$. Then there exist distinct $x_1, \dots, x_n \in b$, which by the Hausdorff axiom can be separated by disjoint relatively open $U_1, \dots, U_n \subseteq a$. Then $\diamond U_1 \cap \dots \cap \diamond U_n \cap \square a$ is a relatively open neighborhood of b disjoint from $\square_{<n}a$.

Now let $b, c \in \square_{<n}a$ be distinct sets. Wlog assume that there is a point $x \in b \setminus c$. Since c is finite and a satisfies the Hausdorff axiom, there is a relatively open superset U of c and a relatively open $V \ni x$, such that $U \cap V = \emptyset$. Now $\diamond V \cap \square_{<n}a$ is a neighborhood of b and $\square U \cap \square_{<n}a$ is a neighborhood of c in $\square_{<n}a$, and they are disjoint. Hence $\square_{<n}a$ is Hausdorff.

(2): It suffices to prove that $a \times a$ is a set and carries at least the product topology, because then it follows inductively that this is also true for a^n with $n \geq 2$. And from this in turn it follows that $a_1 \times \dots \times a_n$ is closed in a^n and carries the subset topology, which implies the claim.

Since a^2 contains exactly the sets of the form $\{\{x\}, \{x, y\}\}$ with $x, y \in a$, it is a subclass of the set $s = \square_{\leq 2}a \cap \diamond \square_{\leq 1}a$ and we only have to prove that it is closed in s . So let $c \in s \setminus a^2$. Then $c = \{\{x\}, \{y, z\}\}$ with $x \notin \{y, z\}$ and $x, y, z \in a$. Since a is Hausdorff, there are disjoint $U \ni x$ and $V \ni y, z$ which are relatively open in a . Then $s \cap \diamond \square_{\leq 1}U \cap \diamond \square_{\leq 2}V$ is relatively open in s , and is a neighborhood of c disjoint from a^2 .

It remains to prove the claim about the product topology, that is, that for every subset $b \subseteq a$, $b \times a$ and $a \times b$ are closed, too. The first one is easy, because $b \times a = a^2 \cap \diamond \square_{\leq 1}b$. Similarly, $(b \times a) \cup (a \times b) = a^2 \cap \diamond \diamond b$, so in order to show that $a \times b$ is closed, let $c \in (b \times a) \cup (a \times b) \setminus (a \times b)$, that is, $c = \{\{x\}, \{x, y\}\}$ with $y \notin b$ and $x \in b$. Since a is Hausdorff, there are relatively open disjoint subsets $U \ni x$ and $V \ni y$ of a . Then $s \cap \diamond \square_{\leq 1}U \cap \diamond \diamond (V \setminus b)$ is a relatively open neighborhood of c disjoint from $a \times b$.

(3): Let $F : a_1 \rightarrow a_2$ be continuous and $\langle x, y \rangle \in a_1 \times a_2 \setminus F$, that is, $F(x) \neq y$. Then $F(x)$ and y can be separated by relatively open subsets $U \ni F(x)$ and $V \ni y$ of a_2 , and since F is continuous, $F^{-1}[U]$ is relatively open in a_1 . $F^{-1}[U] \times V$ is a neighborhood of $\langle x, y \rangle$ and disjoint from F . This concludes the proof that F is relatively closed in $a_1 \times a_2$ and hence a set.

(4): Let $F = F_{n,t_1,\dots,t_m}$. Then $F \subseteq a^n \times a^m \in \mathbb{V}$, so we only have to find for every

$$b = \langle \langle x_1, \dots, x_n \rangle, \langle y_1, \dots, y_m \rangle \rangle, \text{ such that } x_{t_k} \neq y_k \text{ for some } k,$$

a neighborhood disjoint from F . By the Hausdorff property, there are disjoint relatively open $U \ni x_{t_k}$ and $V \ni y_k$. Then

$$(a^{t_k-1} \times U \times a^{n-t_k}) \times (a^{k-1} \times V \times a^{m-k})$$

is such a neighborhood.

(5): Firstly,

$$F \cup (a' \times b) = \bigcap_{x \in a_1} (\{\langle x, F(x) \rangle\} \cup ((a \setminus \{x\}) \times b))$$

is a set for any such function F .

Secondly, the claim about the product topology follows as soon as we have demonstrated the product to be \mathbb{T} -closed, because the product topology is generated by classes of the form $\prod_{x \in a_1} c_x$, where $c_x \subseteq b_x$ and only finitely many c_x differ from b_x .

Since $a \times b$ is \mathbb{T} -closed and the product $P = \prod_{x \in a_1} b_x$ is a subset of $\square(a \times b)$, it suffices to show that P is relatively closed in $\square(a \times b)$, so let $r \in \square(a \times b) \setminus P$. There are four cases:

- The domain of r is not a . Then there is an $x \in a$ such that $x \notin \text{dom}(r)$. In that case, $\square(a \times b) \cap \diamond(\{x\} \times b)$ is a closed superset of P omitting r .
- $a' \times b \not\subseteq r$. Then some $\langle x, y \rangle \in a' \times b$ is missing and $\square(a \times b) \cap \diamond(\{x, y\})$ is a corresponding superset of P .
- $r \upharpoonright a_1$ is not a function. Then there is an $x \in a_1$, such that there exist distinct $\langle x, y_0 \rangle, \langle x, y_1 \rangle \in r$. Since b is Hausdorff, there are closed $u_0, u_1 \subseteq b$, such that $u_0 \cup u_1 = b$, $y_0 \notin u_0$ and $y_1 \notin u_1$. Then P is a subclass of

$$\square(\{x\} \times u_0 \cup (a \setminus \{x\}) \times b) \cup \square(\{x\} \times u_1 \cup (a \setminus \{x\}) \times b),$$

which does not contain r .

- $F = r \upharpoonright a_1$ is a function, but $F(x) \notin b_x$ for some $x \in a_1$. Then

$$\square(\{x\} \times b_x \cup (a \setminus \{x\}) \times b)$$

is a closed superclass of P omitting r .

Thus for every $r \in \square(a \times b) \setminus P$, there is a closed superclass of P which does not contain r . Therefore P is closed. \square

The additivity axiom states that the universe is \mathcal{D} -additive, that is, that the union of a discrete set's image is \mathbb{T} -closed. In other words: For every function F whose domain is a discrete set, the union of the range $\bigcup \text{rng}(F)$ is a set or empty. Had we opted against proper classes, the additivity axiom therefore could have been expressed as an axiom scheme.

Even without a choice principle, we could equivalently have used injective functions into discrete sets instead of surjective functions defined on discrete sets: Point (2) in the following proposition is exactly the additivity axiom.

Proposition 2. *In ES without the additivity axiom, the following are equivalent:*

1. *Images of discrete sets are sets, and unions of discrete sets are \mathbb{T} -closed.*
2. *If d is discrete and $F : d \rightarrow A$ surjective, then $\bigcup A$ is \mathbb{T} -closed.*
3. *If d is discrete and $F : A \hookrightarrow d$ injective, then $\bigcup A$ is \mathbb{T} -closed.*

Proof. (1) \Rightarrow (2): If images of discrete sets are sets, then they are discrete, too, because all their subsets are images of subsets of a discrete set. Thus $F[d]$ is discrete, and therefore its union $\bigcup F[d]$ is closed.

(2) \Rightarrow (1): If d is discrete and F is a function, consider the function $G : \text{dom}(F) \rightarrow \mathbb{V}$ defined by $G(x) = \{F(x)\}$. Then $F[d] = \bigcup G[d] \in \mathbb{V}$. Applying (2) to the identity proves that $\bigcup d = \bigcup \text{id}[d]$ is closed.

(2) \Rightarrow (3): If $F : A \hookrightarrow d$ is an injection, then $F^{-1} : F[A] \rightarrow A$ is a surjection from the discrete subset $F[A] \subseteq d$ onto A , so $\bigcup A$ is closed.

(3) \Rightarrow (2): First we show that $\Box d$ is discrete. We have to show that any given $a \in \Box d$ is not an accumulation point, i.e. that $\Box d \setminus \{a\}$ is closed. Since a is a discrete set, every $d \setminus \{b\}$ for $b \in a$ is closed, as well as $d \setminus a$. But

$$\Box d \setminus \{a\} = \Box d \cap \left(\Diamond(d \setminus a) \cup \bigcup_{b \in a} \Box(d \setminus \{b\}) \right)$$

and this union can be seen to be closed by applying (3) to the map

$$F : \{\Box(d \setminus \{b\}) \mid b \in a\} \hookrightarrow a, \quad F(\Box(d \setminus \{b\})) = b.$$

Now we can prove (2):

Let $G : d \rightarrow \mathbb{V}$. Then $F : G[d] \rightarrow \Box d, F(x) = G^{-1}[\{x\}]$ is an injective function from $G[d]$ to the discrete set $\Box d$. Therefore, $\bigcup G[d] \in \mathbb{V}$. \square

Proposition 3 (ES). $\Box d$ is discrete for every discrete set d . Every \mathcal{D} -small nonempty class is a discrete set and every nonempty union of \mathcal{D} -few discrete sets is a discrete set.

Proof. The first claim has already been shown in the proof of Proposition 2.

Let A be \mathcal{D} -small and $B \subseteq A$. Then B and $\tilde{B} = \{\{b\} \mid b \in B\}$ are also \mathcal{D} -small. Therefore $\bigcup \tilde{B} = B$ is \mathbb{T} -closed by the additivity axiom.

Finally, let A be \mathcal{D} -small and let every $a \in A$ be a discrete set. We have to show that every nonempty $B \subseteq \bigcup A$ is a set. But if A is \mathcal{D} -small, the class C of all nonempty sets of the form $B \cap a$ with $a \in A$ also is. Since $B \neq \emptyset$ and every $a \in A$ is discrete, the union of C is in fact B . \square

2 Ordinal Numbers

We do not assume that the empty class is a set, so there may be no well-founded sets at all, yet of course we want to define the natural numbers and later we will even be looking for an interpretation of a well-founded theory. To this end we need suitable variants of the concepts of well-foundedness and von Neumann ordinal numbers.

Our starting point is finding a substitute for the empty set: A class or atom 0 is called a *zero* if no element of 0 is a superset of 0 . Zeros exist in \mathbb{V} : By the nontriviality axiom, there are distinct $x, y \in \mathbb{V}$, so we can set $0 = \{\{x\}, \{y\}\}$. But in many interesting cases, there even is a definable zero: Let us set $0 = \emptyset$ if $\emptyset \in \mathbb{V}$, and if $\emptyset \notin \mathbb{V}$ but $\mathbb{V} \in \mathbb{V}$, we set $0 = \{\{\mathbb{V}\}\}$ (its element $\{\mathbb{V}\}$ is not a superset of 0 , because by the nontriviality axiom \mathbb{V} is not a singleton). Note that all these examples are sets with at most two elements.

Given a fixed zero 0 , we make the following definitions:

$$\begin{aligned} A^\oplus &= A \setminus 0 \\ A \in_0 B &\text{ if } A \in B^\oplus \text{ and } 0 \subseteq B. \\ A \text{ is } 0\text{-transitive} &\text{ if } c \in_0 A \text{ for all } c \in_0 b \in_0 A. \\ A \text{ } 0\text{-transitive } a \text{ is } 0\text{-pristine} &\text{ if } 0 \subseteq c \notin A \text{ for all } c \in_0 a \cup \{a\}. \\ \alpha \text{ is a } 0\text{-ordinal number} &\text{ if } \alpha \text{ is } 0\text{-transitive, } 0\text{-pristine and} \\ &\alpha^\oplus \text{ is strictly well-ordered by } \in_0, \end{aligned}$$

where by a (strict) *well-order* we mean a (strict) linear order such that each nonempty subset has a minimal element. A (strict) order with the property that every subclass has a minimal element is called a (strict) *strong well-order*, and we will see shortly that in fact such α^\oplus are strictly strongly well-ordered.

We denote the class of 0 -ordinals by On_0 and the 0 -ordinals themselves by lowercase greek letters. If α and β are 0 -ordinals, we also write $\alpha \leq_0 \beta$ for $\alpha \subseteq \beta$. A 0 -ordinal $\alpha \neq 0$ is a *0-limit*

ordinal if it is not the immediate \leq_0 -successor of another 0-ordinal, and it is a 0-cardinal number if there is no surjective map from β^\oplus onto α^\oplus for any $\beta <_0 \alpha$. If there is a least 0-limit ordinal distinct from 0 itself, we call it ω_0 , otherwise we define $\omega_0 = On_0$. Its predecessors $n \in_0 \omega_0$ are the 0-natural numbers. Obviously 0 is the least 0-ordinal, if $0 \in \mathbb{V}$.

For the remainder of this section, let us assume that our 0 is an atom or a finite set. Unless there is danger of confusion (as in the case of \in_0), we omit the prefix and index 0.

Proposition 4 (ES). *Let $\alpha \in On$.*

1. $\alpha \notin \alpha$, α is discrete and $\alpha = 0 \cup \{\beta \in On \mid \beta \in_0 \alpha\}$.
2. On is strictly strongly well-ordered by \in_0 and $<$, and these orders coincide.
3. $\alpha \cup \{\alpha\}$ is the unique immediate successor of α .
4. If A is a nonempty class of ordinals and $\bigcup A \in \mathbb{V}$, then $\bigcup A$ is an ordinal and the least upper bound of A .
5. $\bigcup On = On \cup 0 \notin \mathbb{V}$

Proof. (1): Since $0 \subseteq \alpha$, the equality follows if we can prove that every $x \in_0 \alpha$ is an ordinal. Firstly, let $c \in_0 b \in_0 x$. Then $b \in_0 \alpha$ and $c \in_0 \alpha$ by transitivity of α . Since α^\oplus is strictly linearly ordered by \in_0 , it follows that $c \in_0 x$, proving that x is transitive. Again by the transitivity of α , we see that $x \subseteq \alpha$, and as a subset of a well-ordered set, x^\oplus is well-ordered itself. Also, every $c \in_0 x \cup \{x\}$ is an element of α^\oplus and therefore a superset of 0 not in \mathbb{A} , so x is pristine.

Since α is a superset of 0, $\alpha \notin 0$. Thus if α were an element of α , it would be in α^\oplus . But $\alpha \in_0 \alpha$ contradicts the condition that the elements of α^\oplus are strictly well-ordered.

Because 0 is a discrete set and $\alpha^\oplus = \{x \in \alpha \mid 0 \subseteq x \subseteq \alpha\}$ is closed, it suffices to show that α^\oplus is discrete. So let $\gamma \in_0 \alpha$. Since the elements of α^\oplus are strictly linearly ordered, every $\delta \in \alpha^\oplus \setminus \{\gamma\}$ is either a predecessor or a successor of γ . Hence

$$\alpha^\oplus \setminus \{\gamma\} = \gamma^\oplus \cup \{x \in \alpha^\oplus \mid \{\gamma\} \subseteq x \subseteq \alpha\}$$

is closed.

(2): If $\alpha \in_0 \beta$, then by transitivity of β , α is a subset of β and because $\alpha \notin_0 \alpha$, it is a proper one. For the converse assume $\alpha < \beta$, that is, $\alpha \subset \beta$. β^\oplus is discrete and well-ordered, so the nonempty subset $\beta \setminus \alpha$ contains a minimal element δ , which by (1) is an ordinal number. For all $\gamma \in_0 \delta$, it follows from the minimality of δ that $\gamma \in_0 \alpha$. Now let $\gamma \in_0 \alpha$. Then $\gamma \in_0 \beta$ and since β is linearly ordered, γ is comparable with δ . But if $\delta \in_0 \gamma$, then $\delta \in_0 \alpha$ by transitivity, which is false. Hence $\gamma \in_0 \delta$. We have shown that δ and α have the same predecessors, so by (1), they are equal. Thus $\alpha = \delta \in_0 \beta$ and so the orders \in_0 and $<$ coincide on the ordinals.

Next we show that ordinals $\alpha, \beta \in On$ are always subsets of each other and hence On is linearly ordered, so assume they are not. Let α_0 be minimal in $\alpha \setminus \beta$ and β_0 in $\beta \setminus \alpha$. Now all predecessors of α_0 must be in $\alpha \cap \beta$. And since α and β are transitive, $\alpha \cap \beta$ is an initial segment and therefore every element of $\alpha \cap \beta$ is also in α_0 . The same argument applied to β_0 shows that $\alpha_0 = \alpha \cap \beta = \beta_0$, contradicting our assumption.

Finally, given a nonempty subclass $A \subseteq On$, let $\alpha \in A$ be arbitrary. Then either α has no predecessor in A and thus is minimal itself, or $\alpha \cap A$ is nonempty and has a minimal element δ , because α^\oplus is well-ordered and discrete and $\alpha \cap A \subseteq \alpha^\oplus$. For every $\gamma \in A \setminus \alpha$, we then have $\delta < \alpha \leq \gamma$. Hence δ is in fact minimal in A , concluding the proof that On is strongly well-ordered.

(3): First we verify that $\beta = \alpha \cup \{\alpha\}$ is an ordinal. Since α is transitive, β also is. Since α is pristine and $0 \subseteq \beta \notin \mathbb{A}$, β is pristine itself. And β^\oplus is a set of ordinal numbers, which by (2) must be well-ordered.

From $\alpha \notin \alpha$ it follows that in fact $\beta \neq \alpha$ and thus $\beta > \alpha$. If $\gamma < \beta$, then $\gamma \in_0 \beta$, so either $\gamma \in_0 \alpha$ or $\gamma = \alpha$, which shows that β is an immediate successor. Since the ordinals are linearly ordered, it is the only one.

(4): As a union of transitive, pristine, well-founded sets, $\bigcup A$ is transitive, pristine and well-founded itself. Since all its predecessors are ordinals, they are strictly well-ordered by (2), so it

is an ordinal itself. For each $\beta \in A$, $\beta \subseteq \bigcup A$ and thus $\beta \leq \bigcup A$, so it is an upper bound of A . If $\beta < \bigcup A$, there is an element $\gamma \in A$ with $\beta < \gamma$, therefore it is the least upper bound.

(5): By (1), every element x of an ordinal is in $0 \cup On$. Conversely, 0 is an ordinal and by (3), every ordinal is an element of its successor. Therefore, $0 \cup On = \bigcup On$. If $\bigcup On$ were a set, so would $\bigcup On \cup \{\bigcup On\}$ be. But by (4), that would be an ordinal strictly greater than all elements of On , which is a contradiction. \square

These features of On are all quite desirable, and familiar from Zermelo-Fraenkel set theory. Just as in ZF, On (or rather $On \cup 0$) resembles an ordinal number itself, except that it is not a set. But in ZF, On even has the properties of a regular limit cardinal – a consequence of the replacement axiom. Also, our dependence on the choice of a specific set 0 is rather irritating. This is where the additivity axiom comes in. In the context of ordinal numbers (and discrete sets in general), it is the appropriate analog to the replacement axiom.

By the usual argument, all strongly well-ordered classes whose initial segments are discrete sets are comparable with respect to their length: There is always a unique isomorphism from one of them to an initial segment of the other. In particular, for all finite zeros $0, \bar{0} \in \mathbb{V}$, the well-orders of On_0 and $On_{\bar{0}}$ are comparable. But if $A \subseteq On_0$ is an initial segment isomorphic to $On_{\bar{0}}$, then in fact $A = On_0$, because otherwise A would be a discrete set and by the additivity axiom, $On_{\bar{0}} \in \mathbb{V}$, a contradiction. Hence On_0 and $On_{\bar{0}}$ are in fact isomorphic and the choice of 0 is not relevant to our theory of ordinal numbers. Also, ω_0 and $\omega_{\bar{0}}$ are equally long and we can define a class A to be *finite* if there is a bijection from \mathfrak{n}^{\oplus} to A for some natural number \mathfrak{n} . Otherwise it is *infinite*. It is easy to prove that this definition is equivalent to A being the image of some \mathfrak{n}^{\oplus} or embeddable into some \mathfrak{n}^{\oplus} . Also, it can be stated without quantifying over classes, because such a bijection is defined on a discrete set and therefore a discrete set itself.

Even if there is no limit ordinal, there might still be infinite sets – they just cannot be discrete. So the proper axiom of infinity in the context of essential set theory is the existence of a limit ordinal number:

$$\text{Infinity} \quad \omega \in \mathbb{V}$$

We add the axiom of infinity to a theory by indexing it with the symbol ∞ .

Using induction on ordinal numbers, one easily proves that for each $\alpha \in On$, the least ordinal $\kappa \in On$ such that there is a surjection from κ^{\oplus} to α^{\oplus} is a cardinal, and there is a bijection from κ^{\oplus} to α^{\oplus} .

Proposition 5 (ES). *On is a regular limit, that is:*

1. Every function $F : \alpha^{\oplus} \rightarrow On$ is bounded.
2. The class of cardinal numbers is unbounded in On .

Proof. (1): By the additivity axiom, $\bigcup F[\alpha^{\oplus}]$ is a discrete set, so by Proposition 4, it is an ordinal number and an upper bound of $F[\alpha^{\oplus}]$.

(2): Let us show that for each α there exists a cardinal $\mathfrak{v} > \alpha$. This goes by the usual argument: Every well-order $R \subseteq \alpha^{\oplus} \times \alpha^{\oplus}$ on a subset of α^{\oplus} is a subclass of the discrete set $\square\square\alpha^{\oplus}$, so it is a set itself and since α^{\oplus} is discrete, it is even a strong well-order. Recursively, isomorphisms from initial segments of α^{\oplus} with respect to R to initial segments of On can be defined, and their union is a function from α^{\oplus} onto some β^{\oplus} . We call β the *order type* of R . Now the class A of all well-orders of α is a subclass of $\square\square\square\alpha$ and hence also a discrete set. Mapping every element of A to its order type must therefore define a bounded map $F : A \rightarrow On$. Let $\mathfrak{v} = \min(On \setminus \bigcup F[A])$ be the least ordinal which is not an order type of any subset of α^{\oplus} . We show that \mathfrak{v} is a cardinal above α . Firstly, α is the order type of a well-order of α^{\oplus} , so $\mathfrak{v} > \alpha$. Secondly, assume that $g : \gamma^{\oplus} \rightarrow \mathfrak{v}^{\oplus}$ is surjective and $\gamma < \mathfrak{v}$. Then this defines a well-order on γ^{\oplus} of order-type at least \mathfrak{v} , and since γ is the order type of a well-order on some subset of α^{\oplus} by definition, g would define a well-order on a subset of α^{\oplus} of order-type \mathfrak{v} , a contradiction. \square

If $\mathbb{V} \notin \mathbb{V}$, the closure of On may well be all of \mathbb{V} and in particular does not have to be a set. But in the case $\mathbb{V} \in \mathbb{V}$, the fact that all $\diamond a$ are sets determines the closure Ω of $0 \cup On = \bigcup On$ much

more precisely. Moreover, On then resembles a weakly compact cardinal, which will in fact turn out to be crucial for the consistency strength of the axiom $\mathbb{V} \in \mathbb{V}$.

- Proposition 6 (TS).** 1. Every sequence $\langle x_\alpha \mid \alpha \in On \rangle$ of length On has an accumulation point.
2. Every monotonously \subseteq -decreasing sequence $\langle x_\alpha \mid \alpha \in On \rangle$ of nonempty sets converges to $\bigcap_{\alpha \in On} x_\alpha$.
And every monotonously \subseteq -increasing one to $\text{cl}(\bigcup_{\alpha \in On} x_\alpha)$.
3. $\Omega = 0 \cup On \cup \{\Omega\}$
4. $P = \{x \mid 0 \cup \{\Omega\} \subseteq x \subseteq \Omega\}$ is a perfect set, that is, $P' = P \neq \emptyset$.
5. On has the tree property, that is: If

$$T \subseteq \{f : \alpha^\oplus \rightarrow \mathbb{V} \mid \alpha \in On\}$$

such that $T_\alpha = \{f \upharpoonright \alpha^\oplus \mid f \in T, \alpha^\oplus \subseteq \text{dom}(f)\}$ is discrete and nonempty for each ordinal $\alpha > 0$, then there is a $G : On \rightarrow \mathbb{V}$ such that:

$$G \upharpoonright \alpha^\oplus \in T_\alpha \quad \text{for every } \alpha \in On.$$

Proof. (1): Assume that there is no accumulation point. Then every point $y \in \mathbb{V}$ has a neighborhood U such that $\{\alpha \mid x_\alpha \in U\}$ is bounded in On and therefore discrete. Since the class $\{x_\alpha \mid x_\alpha \in U\}$ of members in U is the image of $\{\alpha \mid x_\alpha \in U\}$, it is also a discrete set and does not have y as its accumulation point. It follows that firstly, $\{\alpha \mid x_\alpha = y\}$ is discrete for each y , and secondly, the image $\{x_\alpha \mid \alpha \in On\}$ of the sequence is also discrete. But On is the union of the sets $\{\alpha \mid x_\alpha = y\}$ for $y \in \{x_\alpha \mid \alpha \in On\}$. Since \mathcal{D} -small unions of discrete sets are discrete sets, this would imply that On is a discrete set, a contradiction.

(2): First let the sequence be decreasing. Then for every $y \in \bigcap_\alpha x_\alpha$, every member of the sequence lies in the closed set $\Diamond\{y\}$, so all its accumulation points do. Now let $y \notin \bigcap_\alpha x_\alpha$. Then there is a $\beta \in On$ such that $y \notin x_\beta$, and hence from x_β on, all members are in $\Box x_\beta$, so all accumulation points are. Thus the only accumulation point is the intersection. (Note that the intersection therefore is nonempty because $\Diamond\mathbb{V}$ is a closed set containing every member of the sequence.)

Now assume that the sequence is ascending and let A be its union. If $y \in A$, then $y \in x_\beta$ for some $\beta \in On$. Then all members from x_β on are in $\Diamond\{y\}$, so each accumulation point also is. Thus all accumulation points are supersets of A . But all members of the sequence are in $\Box \text{cl}(A)$, so each accumulation point is a subset of $\text{cl}(A)$, and therefore equal to $\text{cl}(A)$.

(3): It suffices to prove that Ω is the unique accumulation point of On . Since On is the image of an increasing sequence, its accumulation point is indeed unique and is the closure of $\bigcup On$ by (2). But $\text{cl}(\bigcup On) = \text{cl}(0 \cup On) = \Omega$.

(4): P is closed, and it is nonempty because $\Omega \in P$. Given $x \in P$, the sequences in P given by

$$y_\alpha = x \setminus (On \setminus \alpha^\oplus) \quad \text{and} \quad z_\alpha = x \cup (On \setminus \alpha^\oplus)$$

both converge to x by (2). If $x \cap On$ is unbounded, x is not among the y_α , otherwise it is not among the z_α , so in any case, x is the limit of a nontrivial sequence in P .

(5): Since for every $\alpha \in On$, T_α is nonempty, there is for every α an $f \in T$ with $\alpha^\oplus \subseteq \text{dom}(f)$. Thus the map

$$T \rightarrow On, \quad f \mapsto 0 \cup \text{dom}(f)$$

is unbounded in On and therefore has a nondiscrete image. Hence T is not discrete and has an accumulation point $g \in \mathbb{V}$. We set $G = g \cap (On \times \mathbb{V})$.

For each $\alpha \in On$, the union $\bigcup_{\beta < \alpha} T_\beta$ is a discrete set, so g is an accumulation point of the difference $T \setminus \bigcup_{\beta < \alpha} T_\beta$, which is the class of all those $f \in T$ whose domain is at least α^\oplus . But every such f is by definition the extension of some $h \in T_\alpha$. Thus this difference is the union of the classes $S_h = \{f \in T \mid h \subseteq f\}$ with $h \in T_\alpha$. Since T_α is discrete, $\text{cl}(\bigcup_{h \in T_\alpha} S_h) = \bigcup_{h \in T_\alpha} \text{cl}(S_h)$, so g must be in the closure of some S_h . But S_h is a subclass of the closed

$$\{x \mid h \subseteq x \subseteq h \cup (\Omega \setminus \alpha \times \mathbb{V})\},$$

so $h \subseteq g \subseteq h \cup (\Omega \setminus \alpha \times \mathbb{V})$, too.

We have shown that for every $\alpha \in On$, the set $g \cap (\alpha^\oplus \times \mathbb{V})$ is an element of T_α . This implies that G is a function defined on On , and that $G \restriction \alpha^\oplus = f \restriction \alpha^\oplus$ for some $f \in T$, concluding the proof. \square

In fact, we have just shown that every accumulation point g of T gives rise to such a solution G . Hence firstly, $T = \text{cl}(T) \cap \square(On \times \mathbb{V})$, and secondly, G can always be described as the intersection of a set g with $On \times \mathbb{V}$. In our formulation of the tree property, the two quantifications over classes could thus be replaced by quantifications over sets.

If Ω exists, the hierarchy of well-ordered sets extends well beyond the realm of ordinal numbers. By *linearly ordered set* we shall mean from now on a set together with a linear order \leq such that the set's natural topology is at least as fine as the order topology, that is, such that all \leq -closed intervals are \mathbb{T} -closed. And by *well-ordered set* we mean a linearly ordered set whose order is a well-order (or a strong well-order – which in this case is equivalent). Then all well-ordered sets are comparable.

The significance of (4) is that even if $\mathbb{V} \in \mathbb{V}$, the universe cannot be a well-ordered set, because well-ordered sets have no perfect subset. Thus whenever a is a well-ordered set, there is a $p \notin a$ and the set $a \cup \{p\}$ can be well-ordered such that its order-type is the successor of the order-type of a . We use the usual notation for intervals in the context of linearly-ordered sets, and consider ∞ (resp. $-\infty$) as greater (resp. smaller) than all the elements of the set. We will also sloppily write $a + b$ and $a \cdot b$ for order-theoretic sums and products and say that an order-type *exists* if there is a linearly ordered set with that order-type.

Every linearly ordered set a is a Hausdorff set and since its order is closed with respect to the product topology, it is itself a set by Proposition 1. Moreover, the class

$$\bigcap_{b \subseteq a \text{ initial segment}} \square b \cup \{c \mid b \subseteq c \subseteq a\}$$

of all its \mathbb{T} -closed initial segments is itself a linearly ordered set in which a can be embedded via $x \mapsto (-\infty, x]$. Thus we can limit our investigations to well-ordered sets whose order is given by \subseteq and whose union exists, which makes things considerably easier:

Lemma 7 (ES). *If a class $A \subseteq \square a$ is linearly ordered by \subseteq , then $\text{cl}(A)$ is a linearly ordered set ordered by \subseteq . If A is well-ordered, then so is $\text{cl}(A)$.*

Proof. First we prove that $\text{cl}(A)$ is still linearly ordered. Let $x, y \in \text{cl}(A)$ and assume that $x \not\subseteq y$. Every $z \in A$ is comparable to every other element of A , so A is a subclass of the set $\square z \cup \{v \mid z \subseteq v \subseteq a\}$ and thus $\text{cl}(A)$ also is. Therefore both x and y are comparable to every element of A and A is a subclass of $\square x \cup \{v \mid x \subseteq v \subseteq a\}$. Since y is not a superset of x , it must be in the closure of $A \cap \square x$ and thus a subset of x .

Since $\{v \in \text{cl}(A) \mid x \subseteq v \subseteq y\} = [x, y]$ is closed, $\text{cl}(A)$ in fact carries at least the order topology.

Now assume that A is well-ordered and let $B \subseteq \text{cl}(A)$ be nonempty. Wlog let B be a final segment. If B has only one element, then that element is minimal, so assume it has at least two distinct elements. Since A is dense, it must then intersect B and $A \cap B$ must have a minimal element x . Assume that x is not minimal in B . Then there is a $y \subset x$ in $B \setminus A$, and this y must be minimal, because if there were a $z \subset y$ in B , then (z, x) would be a nonempty open interval in $\text{cl}(A)$ disjoint from A . \square

Thanks to this lemma, to prove that well-ordered sets of a certain length exist, it suffices to give a corresponding subclass of some $\square a$ well-ordered by \subseteq . As the next theorem shows, this enables us to do a great deal of well-order arithmetic in essential set theory.

Proposition 8 (ES). *If a and b are Hausdorff sets and $a_x \subseteq a$ is a well-ordered set for every $x \in b_1$, then $\sup_{x \in b_1} a_x$ exists. If in addition, R is a well-order on b_1 (not necessarily a set), then $\sum_{x \in b_1} a_x$ exists. In particular, the order-type of R exists, and binary sums and products of well-orders exist.*

Proof. Consider families $\langle r_x \mid x \in b_1 \rangle$ of initial segments $r_x \subseteq a_x$ with the following property: for all $x, y \in b_1$ such that $r_x \neq a_x$, the length of r_y is the maximum of r_x and a_y . Given such a

family, the class

$$b' \times a \cup \bigcup_{x \in b_I} \{x\} \times r_x$$

is a set. And the class of all such sets is a subclass of $\square(b \times a)$ well-ordered by \subseteq and at least as long as every a_x , because assigning to $y \in a_x$ the set

$$b' \times a \cup \bigcup_{z \in b_I} \{z\} \times r_z,$$

is an order-preserving map, where $r_z = a_z$ whenever a_z is at most as long as a_x , and $r_z = (-\infty, \tilde{y}]$ such that r_z is order-isomorphic to $(-\infty, y]$ otherwise.

In the well-ordered case, consider for every $\langle x, y \rangle \in b_I \times a$ with $y \in a_x$ the set

$$b' \times a \cup \{x\} \times (-\infty, a_x] \cup (-\infty, x)_R \times a.$$

The class of these sets is again a subclass of $\square(b \times a)$ and well-ordered by \subseteq . Its order-type is the sum of the orders a_x .

Setting $a_x = 1^\oplus$ for each x yields a well-ordered set of the length of R . Using a two-point b proves that binary sums exist. And if b is a well-ordered set and $a_x = a$ for each $x \in b_I$, then $(b + 1^\oplus)_I$ has at least the length of b and $a \cdot b$ can be embedded in $\sum_{x \in (b + 1^\oplus)_I} a_x$. \square

3 Pristine Sets and Inner Models

Pristine sets are not only useful for obtaining ordinal numbers, but also provide a rich class of inner models of essential set theory and prove several relative consistency results. To this end, we need to generalize the notion of a pristine set, such that it also applies to non-transitive sets.

But first we give a general criterion for interpretations of essential set theory. The picture behind the following is this: The elements of the class Z are to be ignored, so Z is interpreted as the empty class. We do this to be able to interpret $\emptyset \in \mathbb{V}$ even if the empty class is proper by choosing a nonempty set $Z \in \mathbb{V}$. Everything that is to be interpreted as a class will be a superclass X of Z , but only the elements of $X \setminus Z$ correspond to actual objects of the interpretation. In particular, $B \supseteq Z$ will be interpreted as the class of atoms and W as the universe. So the extension of an element $x \in W \setminus B$ will be a set X with $Z \subseteq X \subseteq W$, which we denote by $\Phi(x)$. Theorem 9 details the requirements these objects must meet to define an interpretation of ES.

Theorem 9 (ES). *Let $\mathcal{K} \subseteq \mathcal{D}$ and $Z \subseteq B \subseteq W$ be classes and $\Phi : W \setminus B \rightarrow \mathbb{V}$ injective. We use the following notation:*

- X is an inner class if it is not an atom and $Z \subseteq X \subseteq W$. In that case, let $X^\oplus = X \setminus Z$.
- $S = W \setminus B^\oplus$ and $T = \Phi[S^\oplus]$.
- $\overline{\Phi} = \Phi \cup \text{id}_{B^\oplus} : W^\oplus \rightarrow \mathbb{V}$

Define an interpretation \mathcal{I} as follows:

$$\begin{aligned} X \text{ is in the domain of } \mathcal{I} & \quad \text{if} \quad X \text{ is an inner class or } X \in B^\oplus. \\ X \in^{\mathcal{I}} Y & \quad \text{if} \quad Y \text{ is an inner class and } X \in \overline{\Phi}[Y^\oplus]. \\ A^{\mathcal{I}} & = B \end{aligned}$$

If the following conditions are satisfied, \mathcal{I} interprets essential set theory:

1. W^\oplus has more than one element.
2. Every element of T is an inner class, and no element of B is an inner class.
3. $Z \cup \{x\} \in T$ for every $x \in W^\oplus$.
4. Any intersection $\bigcap C$ of a nonempty $C \subseteq T$ is Z or an element of T .
5. $x \cup y \in T$ for all $x, y \in T$.
6. If $x \in T$ and $x \setminus \{y\} \in T$ for all $y \in x^\oplus$, then x^\oplus is \mathcal{K} -small.

7. Any union $\bigcup C$ of a nonempty \mathcal{K} -small $C \subseteq T$ is an element of T .
8. For all $a, b \in T$, the class $Z \cup \{x \in S^\oplus \mid \Phi(x) \subseteq a, \Phi(x) \cap b \neq Z\}$ is Z or in T .

The length of On^J is the least \mathcal{K} -large ordinal κ , or On if no such κ exists (for example in the case $\mathcal{K} = \mathcal{D}$). In particular, $(\omega \in \mathbb{V})^J$ iff ω is \mathcal{K} -small.

Proof. Let us first translate some J -interpretations of formulas:

- $(X \notin \mathbb{A})^J$ iff X is an inner class, and $(X \in \mathbb{A})^J$ iff $X \in B^\oplus$.
- $(X \in \mathbb{V})^J$ iff $X \in \overline{\Phi}[W^\oplus]$, because W is the union of all inner classes, so $\mathbb{V}^J = W$.
- If $(F : X_1 \rightarrow X_2)^J$, then there is a function $G : \overline{\Phi}[X_1^\oplus] \rightarrow \overline{\Phi}[X_2^\oplus]$, defined by $G(Y_1) = Y_2$ if $(F(Y_1) = Y_2)^J$, and G is surjective resp. injective iff $(F$ is surjective) J resp. $(F$ is injective) J .

Now we verify the axioms of ES^J :

Extensionality: Assume $(X_1 \neq X_2 \wedge X_1, X_2 \notin \mathbb{A})^J$. Then X_1 and X_2 are inner classes. But $X_1 \neq X_2$ implies that there exists an element y in $X_1 \setminus X_2 \subseteq W^\oplus$ or $X_2 \setminus X_1 \subseteq W^\oplus$. $Y = \Phi(y)$ is either in B^\oplus or an inner class by (2). Since Φ is injective, this means by definition that $(Y \in X_1 \wedge Y \notin X_2)^J$ or vice versa.

The *atoms axiom* follows directly from our definition of \in^J , because no element of B^\oplus is an inner class, and we enforced *Nontriviality* by stating that W^\oplus has more than one element.

Comprehension(ψ): If $Y = Z \cup \{x \in W^\oplus \mid \psi^J(\Phi(x), \vec{P})\}$, then Y witnesses the comprehension axiom for the formula $\psi = \phi^C$ with the parameters \vec{P} , because $X \in^J Y$ iff

$$X \in \overline{\Phi}[Y^\oplus] = \{\Phi(x) \mid x \in W^\oplus \wedge \psi^J(\Phi(x), \vec{P})\},$$

which translates to $X \in \overline{\Phi}[W^\oplus]$ and $\psi^J(X, \vec{P})$.

T_1 : Let $(X \in \mathbb{V})^J$. Then $X = \overline{\Phi}(x)$ for some $x \in W^\oplus$. By (3), $Y = Z \cup \{x\} \in T = \Phi[S^\oplus]$, so in particular $(Y \in \mathbb{V})^J$. But X is the unique element such that $X \in^J Y$, so $(Y = \{X\})^J$.

2nd Topology Axiom: Assume $(D$ is a nonempty class of sets) J , because if $(D$ contains an atom) J , the intersection is empty in J anyway. Then D is an inner class and every $Y \in C = \overline{\Phi}[D^\oplus]$ is an inner class, which means $Y \in \Phi[S^\oplus]$. So $C \subseteq \Phi[S^\oplus]$ and $C \neq \emptyset$. We have $(X \in \bigcap D)^J$ iff $X \in^J Y$ for all $Y \in^J D$, that is:

$$X \in \bigcap_{Y \in C} \overline{\Phi}[Y^\oplus] = \overline{\Phi} \left[\left(\bigcap C \right)^\oplus \right],$$

because $\overline{\Phi}$ is injective. Hence the inner class $\bigcap C$ equals $(\bigcap D)^J$, and by (4), it is either in T and therefore interpreted as a set, or it is $Z = \emptyset^J$.

Additivity: A similar argument shows that $\bigcup C$ equals $(\bigcup D)^J$. If $(D$ is a discrete set) J , then by (6), D^\oplus is \mathcal{K} -small and therefore the union of $C = \overline{\Phi}[D^\oplus]$ is in T by (7).

3rd Topology Axiom: Let $(X_1, X_2 \in \mathbb{T})^J$. Then $X_1, X_2 \in T$ and $X_1, X_2 \neq Z$. By (5), $Y = X_1 \cup X_2 \in T$, and Y is interpreted as the union of X_1 and X_2 .

The *Exponential* axiom follows from (8), because

$$Y = Z \cup \{x \in S^\oplus \mid \Phi(x) \subseteq a, \Phi(x) \cap b \neq Z\}$$

equals $(\Box a \cap \Diamond b)^J$. In fact, $X \in^J Y$ iff $X \in T$, $X \subseteq a$ and $X \cap b \neq Z$, and $X \subseteq a$ is equivalent to $(X \subseteq a)^J$, while $X \cap b \neq Z$ is equivalent to $(X \cap b \neq \emptyset)^J$.

The statement about the length of On^J holds true because the discrete sets are interpreted by the classes X with \mathcal{K} -small X^\oplus . \square

All the conditions of the theorem only concern the image of Φ but not Φ itself, so given such a model one can obtain different models by permuting the images of Φ . Also, if $\Phi[S^\oplus]$ is infinite and if $Z \in \mathbb{V}$, one can toggle the truth of the statement $(\emptyset \in \mathbb{V})^J$ by including Z in or removing Z from $\Phi[S^\oplus]$.

Proposition 10 (ES). *If $Z = \emptyset$, T is a \mathcal{K} -compact Hausdorff \mathcal{K} -topology on W , W has at least two elements, $B \subseteq W$ is open and does not contain any subsets of W , and $\Phi : W \setminus B \rightarrow \text{Exp}_{\mathcal{K}}(W, T)$ is a homeomorphism, then all conditions of Theorem 9 are met and therefore these objects define an interpretation of ES. In addition, they interpret the statements $\mathbb{V} \in \mathbb{V}$ and that every set is \mathcal{D} -compact Hausdorff.*

Proof. All conditions that we did not demand explicitly follow immediately from the fact that W is a \mathcal{K} -compact Hausdorff \mathcal{K} -topological space and from the definition of the exponential \mathcal{K} -topology.

$(\mathbb{V} \in \mathbb{V})^J$ holds true, because $W \in \text{Exp}_{\mathcal{K}}(W, T)$. And since the \mathcal{K} -small sets are exactly those interpreted as discrete, the \mathcal{K} -compactness and Hausdorff property of W implies that $(\mathbb{V} \text{ is } \mathcal{D}\text{-compact Hausdorff})^J$. \square

Such a topological space W , together with a homeomorphism Φ to its hyperspace, is called a \mathcal{K} -hyperuniverse. These structures have been extensively studied in [FHL96, FH96b, Ess03]. Here we will deal with a different class of models given by pristine sets.

Let $Z \subseteq B$ be such that no element of B is a superset of Z (they are allowed to be atoms). Again, write $X \in_Z Y$ for:

$$X \in Y^{\oplus} \quad \text{and} \quad Z \subseteq Y.$$

And X is Z -transitive if $c \in_Z X$ whenever $c \in_Z b \in_Z X$. We say that X is Z -B-pristine if:

- $X \in_Z B$ or:
- $Z \subseteq X \notin \mathbb{A}$, and there is a Z -transitive set $b \supseteq X$, such that for every $c \in_Z b$ either $Z \subseteq c \notin \mathbb{A}$ or $c \in_Z B$.

If a has a Z -transitive superset b , then it has a least Z -transitive superset $\text{trcl}(a) = \bigcap \{b \supseteq a \mid b \text{ } Z\text{-transitive}\}$, the Z -transitive closure of a . Obviously a set is Z -transitive iff it equals its Z -transitive closure. Also, a is Z -B-pristine iff $\text{trcl}(a)$ exists and is Z -B-pristine. A set a is Z -well-founded iff for every $b \ni_Z a$, there exists an \in_Z -minimal $c \in_Z b$.

Theorem 11 (ES). *Let $Z \in \mathbb{V}$ and $B \supseteq Z$ such that no element of B is a superset of Z , and B^{\oplus} is \mathbb{T} -closed. Let Φ be the identity on $W \setminus B$ and $\mathcal{K} = \mathcal{D}$. The following classes W_i^{\oplus} meet the requirements of Theorem 9 and therefore define interpretations \mathcal{I}_i of essential set theory:*

- the class W_1^{\oplus} of all Z -B-pristine x
- the class W_2^{\oplus} of all Z -B-pristine x with discrete $\text{trcl}(x)^{\oplus}$
- the class W_3^{\oplus} of all Z -well-founded Z -B-pristine x with discrete $\text{trcl}(x)^{\oplus}$

Z is a member of all three classes and thus $(\emptyset \in \mathbb{V})^{J_1}$ holds true in all three cases. If $i \in \{2, 3\}$, then $(\text{every set is discrete})^{J_i}$, and in the third case, $(\text{every set is } \emptyset\text{-well-founded})^{J_3}$.

If $\mathbb{V} \in \mathbb{V}$, then:

1. $(\mathbb{V} \in \mathbb{V})^{J_1}$
2. $(\text{On has the tree property})_i^J$ for all i .
3. If B^{\oplus} is discrete, \mathcal{I}_3 satisfies the strong comprehension principle.

Proof. In this proof, we will omit the prefixes Z and B : By “pristine” we always mean Z -B-pristine, “transitive” means Z -transitive and “well-founded” Z -well-founded.

Since Z^{\oplus} is empty and B is pristine and well-founded, $Z \in W_3^{\oplus} \subseteq W_2^{\oplus} \subseteq W_1^{\oplus}$.

Before we go through the requirements of Theorem 9, let us prove that x^{\oplus} is closed for every $x \in S^{\oplus}$:

$$x^{\oplus} = (x \cap B^{\oplus}) \cup (\{Z\} \cap x) \cup \{y \in x \mid Z \subseteq y \notin \mathbb{A}\}$$

Since x is pristine, there is a transitive pristine $c \supseteq x$, and we can rewrite the class $\{y \in x \mid Z \subseteq y \notin \mathbb{A}\}$ as $\{y \in x \cap c \mid Z \subseteq y \subseteq c\}$, which is closed.

Condition (1) of Theorem 9 is satisfied because Z and $Z \cup \{Z\}$ are distinct elements of W_3^{\oplus} .

(2): If $x \in B$, then x is not a superset of Z and therefore not an inner class. Now let $x \in S_1^\oplus$. We have to show that $x = \Phi(x)$ is an inner class. Since $x \notin B$ and x is pristine, $x \notin A$ and $Z \subseteq x$, so it only remains to prove that $y \in W_1^\oplus$ for every $y \in x^\oplus$. If $y \in_Z B$, y is pristine. If $y \notin_Z B$, then $Z \subseteq y$. Since every transitive superset of x is also a superset of y , y is pristine in that case, too. If in addition, $\text{trcl}(x)^\oplus$ is discrete, y also has that property, by the same argument. And if x is also well-founded, y also is: For any $b \ni_Z y$, $b^\oplus \cup \{x\}$ has a \in_Z -minimal element; since $y \in_Z x$ and $y \in_Z b$, this cannot be x , so it must be in b^\oplus . This concludes the proof that $y \in W_1^\oplus$ whenever $x \in S_1^\oplus$.

(3): If $x \in W_1^\oplus$, then $Z \cup \{x\}$ is pristine, because if $x \in B^\oplus$, it is already transitive itself, and otherwise if c is a transitive pristine superset of x , then $c \cup \{x\}$ is a transitive pristine superset of $Z \cup \{x\}$. If moreover c^\oplus is discrete, then $c^\oplus \cup \{x\}$ also is, and if x is well-founded, $Z \cup \{x\}$ also is.

(4): Let $C \subseteq S_1^\oplus$ be nonempty. Then $\bigcap C \in S_1^\oplus$, too, because every subset of a pristine set which is a superset of Z is pristine itself, every subset of a discrete set is discrete, and every subset of a well-founded set is well-founded.

(6): Assume that for every $y \in x^\oplus$, we have $x \setminus \{y\} \in S^\oplus$. Then $(x \setminus \{y\})^\oplus = x^\oplus \setminus \{y\}$ is closed, and hence x^\oplus is a discrete set.

(7) (and consequently (5)): Let $C \subseteq S_1^\oplus$ be a nonempty discrete set. Then $\bigcup C \in W_1 \setminus B$, because if c_b is a transitive pristine superset of b for all $b \in C$, then $\bigcup c_b$ is such a superset of the union. If all the c_b are discrete, their union also is, because they are only \mathcal{D} -few. And if every element of C is well-founded, $\bigcup C$ also is.

(8): $Y = Z \cup \{x \in S^\oplus \mid x \subseteq a, x \cap b \neq Z\}$ is pristine, because if c is a transitive pristine superset of a , then $z = Z \cup \{Z\} \cup \{x \in \Box c \mid Z \subseteq x\}$ is a transitive pristine superset of Y . And Y is in fact a set, because b^\oplus is closed, so $Y = Z \cup (z^\oplus \cap \Box a \cap \Diamond b^\oplus)$ also is. If c^\oplus is discrete, $\Box c^\oplus$ is discrete, and so is $z^\oplus \setminus \{Z\} = \{y \cup Z \mid y \in \Box c^\oplus\}$. And if a is well-founded, any set of subsets of a is well-founded, too.

The claims about discreteness and well-foundedness are immediate from the definitions.

Now let us prove the remaining claims under the assumption that $\mathbb{V} \in \mathbb{V}$:

(1): $\mathbb{V} \setminus A$ is a set, namely $\Diamond \mathbb{V} \cup \{\emptyset\}$ or $\Diamond \mathbb{V}$, depending on whether $\emptyset \in \mathbb{V}$. Let:

$$\begin{aligned} U_0 &= \mathbb{V} \\ U_{n+1} &= B^\oplus \cup \{x \in \mathbb{V} \setminus A \mid Z \subseteq x \subseteq Z \cup U_n\} \\ U_\omega &= \bigcap_{n \in \omega} U_n \end{aligned}$$

Then U_ω is a set. Since $W_1^\oplus \subseteq \mathbb{V}$ and $W_1^\oplus \subseteq B^\oplus \cup \{x \in \mathbb{V} \setminus A \mid Z \subseteq x \subseteq Z \cup W_1^\oplus\}$, it is a subset of U_ω . It remains to show that $U_\omega \subseteq W_1^\oplus$, that is, that every element of U_ω is pristine, because then it follows that W_1 is a pristine set itself and hence $W_1 \in_Z W_1$. In fact, it suffices to prove that $Z \cup U_\omega$ is a transitive pristine set, because then all $x \in_Z U_\omega$ will be pristine, too. So assume $y \in_Z x \in_Z Z \cup U_\omega$. If x were in B^\oplus , then $y \notin_Z x$, so x must be in $\mathbb{V} \setminus A$ and $Z \subseteq x \subseteq Z \cup U_n$ for all n . Thus $x \subseteq Z \cup U_\omega$, which implies that $y \in_Z U_\omega$.

(2) follows from Proposition 6.

(3): It suffices to show that W_3^\oplus does not contain any of its accumulation points, because that implies that every inner class corresponds to a set – its closure –, so that the weak comprehension principle allows us to quantify over all inner classes. Since B^\oplus is discrete and

$$S_3^\oplus \setminus \{Z\} = W_3^\oplus \setminus (B^\oplus \cup \{Z\}) \subseteq \{x \in \mathbb{V} \setminus A \mid Z \subseteq x\} \in \mathbb{V}$$

(recall that no element of B is a superset of Z), B certainly contains no accumulation point of W_3^\oplus . So assume now that $x \in W_3^\oplus$ is an accumulation point. Since it is well-founded and $\text{trcl}(x)^\oplus$ is a discrete set, $\text{trcl}(x)^\oplus \cup \{x\}$ has an \in_Z -minimal W_3^\oplus -accumulation point y . Then $y \in S_3^\oplus$ and y is also an accumulation point of $W_3^\oplus \setminus (B^\oplus \cup \{Z\})$. Since none of the \mathcal{D} -few elements of y^\oplus is an W_3^\oplus -accumulation point, $W_3^\oplus \setminus (B^\oplus \cup \{Z, y\})$ is a subclass of

$$\Diamond \text{cl}(W_3^\oplus \setminus y) \cup \bigcup_{z \in_Z y} \Box \text{cl}(W_3^\oplus \setminus \{z\}),$$

which is closed and does not contain y , a contradiction. \square

By the nontriviality axiom, there are distinct $x, y \in \mathbb{V}$. If we set $Z = B = \{\{x\}, \{y\}\}$, the requirements of Theorem 11 are satisfied, so \mathcal{I}_i interprets essential set theory with $\emptyset \in \mathbb{V}$ in all three cases. Moreover, since $Z = B$, it interprets $\mathbb{A} = \emptyset$. So $\mathbb{A} = \emptyset \in \mathbb{V}$ is consistent relative to ES. In the case $i = 3$, moreover, (every set is \emptyset -well-founded and discrete)³! And if in addition $\omega \in \mathbb{V}$, then ω is \mathcal{D} -small and thus $(\omega \in \mathbb{V})^{J_3}$ by Theorem 9.

But if in ES every set is discrete and \emptyset -well-founded, the following statements are implied:

<i>Pair, Union, Power, Empty Set</i>	$\{a, b\}, \bigcup a, \mathfrak{P}(a), \emptyset \in \mathbb{V}$
<i>Replacement</i>	If F is a function and $a \in \mathbb{V}$, then $F[a] \in \mathbb{V}$.
<i>Foundation</i>	Every $x \in \mathbb{T}$ has a member disjoint from itself.

And these are just the axioms of ZF²! Conversely, all the axioms of ES hold true in ZF, so ZF could equivalently be axiomatized as follows³:

- ES_∞
- $\mathbb{A} = \emptyset \in \mathbb{V}$
- Every set is discrete and \emptyset -well-founded.

If in addition $\mathbb{V} \in \mathbb{V}$, then \mathcal{I}_3 even interprets the strong comprehension axiom and therefore Kelley-Morse set theory⁴ with *On* having the tree property. Conversely, O. Esser showed in [Ess97] and [Ess99] that this theory is equiconsistent with GPK_∞^+ , which in turn is an extension of topological set theory that will be introduced in the next section. In summary, we have the following results:

Corollary 12. *ES_∞ is equiconsistent with ZF: The latter implies the former and the former interprets the latter.*

*TS_∞ and GPK_∞^+ both are mutually interpretable with:
Kelley-Morse set theory + *On* has the tree property.*

\mathcal{I}_3 is a particularly intuitive interpretation if $\emptyset, \mathbb{V} \in \mathbb{V}$, $\mathbb{A} = \emptyset$ and we set $Z = B = \emptyset$. Then every set is $(\emptyset\text{-}\emptyset)$ -pristine and \in_N is just \in . Also, $\mathbb{V} \setminus \{\emptyset\} = \Diamond \mathbb{V} \in \mathbb{V}$, so \emptyset is an isolated point. If a set x contains only isolated points, it is discrete, and since $x = \bigcup_{y \in x} \{y\}$ and every $\{y\}$ is open, x is a clopen set. Moreover, x is itself an isolated point, because $\{x\}$ is open:

$$\{x\} = \Box x \cup \bigcup_{y \in x} \Diamond \{y\}$$

Thus it follows that all $(\emptyset\text{-})$ -well-founded sets are isolated. Define the cumulative hierarchy as usual:

$$\begin{aligned} u_0 &= \emptyset \\ u_{\alpha+1} &= \Box u_\alpha \cup \{\emptyset\} \\ u_\lambda &= \bigcup_{\alpha < \lambda} u_\alpha \text{ for limit ordinals } \lambda \end{aligned}$$

Since images of discrete sets in *On* are bounded and since every nonempty class of well-founded sets has an \in -minimal element, the union $\bigcup_{\alpha \in \text{On}} u_\alpha$ is exactly the class of all well-founded sets, and in fact equals W_3 .

²With classes, of course. We avoid the name NBG, because that is usually associated with a strong axiom of choice.

³We will soon introduce a choice principle for ES, the uniformization axiom, which applies to all discrete sets. Since in ZF every set is discrete, that axiom is equivalent to the axiom of choice.

⁴The axiom of choice is not necessarily true in that interpretation, but even the existence of a global choice function does not add to the consistency strength, as was shown in [Ess04].

4 Positive Specification

This section is a short digression from our study of essential set theory. Again starting from only the class axioms we introduce specification schemes for two classes of “positive” formulas as well as O. Esser’s theory GPK^+ (cf. [Ess97, Ess99, Ess00, Ess04]), and then turn our attention to their relationship with topological set theory.

The idea of positive set theory is to weaken the inconsistent *naïve comprehension scheme* – that every class $\{x \mid \phi(x)\}$ is a set – by permitting only *bounded positive formulas* (BPF), which are defined recursively similarly to the set of all formulas, but omitting the negation step, thus avoiding the Russell paradox. This family of formulas can consistently be widened to include all *generalized positive formulas* (GPF), which even allow universal quantification over classes. But to obtain more general results, we will investigate *specification schemes* instead of comprehension schemes, which only state the existence of subclasses $\{x \in c \mid \phi(x)\}$ of sets c . If \mathbb{V} is a set, this restriction makes no difference.

We define recursively when a formula ϕ whose variables are among X_1, X_2, \dots and Y_1, Y_2, \dots (where these variables are all distinct) is a *generalized positive formula* (GPF) with parameters Y_1, Y_2, \dots :

- The atomic formulas $X_i \in X_j$ and $X_i = X_j$ are GPF with parameters Y_1, Y_2, \dots
- If ϕ and ψ are GPF with parameters Y_1, Y_2, \dots , then so are $\phi \wedge \psi$ and $\phi \vee \psi$.
- If $i \neq j$ and ϕ is a GPF with parameters Y_1, Y_2, \dots , then so are $\forall X_i \in X_j \phi$ and $\exists X_i \in X_j \phi$.
- If ϕ is a GPF with parameters Y_1, Y_2, \dots , then so is $\forall X_i \in Y_j \phi$.

A GPF with parameters Y_1, Y_2, \dots is a *bounded positive formula* (BPF) if it does not use any variable Y_i , that is, if it can be constructed without making use of the fourth rule. The *specification axiom* for the GPF $\phi(X_1, \dots, X_m, Y_1, \dots, Y_n)$ with parameters Y_1, Y_2, \dots , whose free variables are among X_1, \dots, X_m , is:

$$\{x \in c \mid \phi(x, b_2, \dots, b_m, B_1, \dots, B_n)\} \text{ is } \mathbb{T}\text{-closed}$$

for all $c, b_2, \dots, b_m \in \mathbb{V}$ and all classes B_1, \dots, B_n .

GPF specification is the scheme consisting of the specification axioms for all GPF ϕ , and *BPF specification* incorporates only those for BPF ϕ . Note that we did not include the formula $x \in \mathbb{A}$ or any other formula involving the constant \mathbb{A} in the definition, so $x \in \mathbb{A}$ is not a GPF.

The following theorem shows that BPF specification is in fact finitely axiomatizable, even without classes.⁵

Theorem 13. *Assume only the class axioms and that for all $a, b \in \mathbb{V}$, the following are \mathbb{T} -closed:*

$$\bigcup a, \quad \{a, b\}, \quad a \times b$$

Let Θ be the statement that for all sets $a, b \in \mathbb{V}$, the following are \mathbb{T} -closed:

$$\Delta na, \quad \mathbf{E}na, \quad \{\langle x, y \rangle \in b \mid \forall z \in y \langle x, y, z \rangle \in a\},$$

$$\{\langle y, x, z \rangle \mid \langle x, y, z \rangle \in a\}, \quad \{\langle z, x, y \rangle \mid \langle x, y, z \rangle \in a\}$$

Then BPF specification is equivalent to Θ . And GPF specification is equivalent to Θ and the second topology axiom.

Proof. Ordered pairs can be built from unordered ones, and the equality $\langle x, y \rangle = z$ can be expressed as a BPF. Therefore the classes mentioned in Θ can all be defined by applying BPF specification to a given set or product of sets, so BPF specification implies Θ .

GPF specification in addition implies the second topology axiom,

$$\forall B \neq \emptyset. \quad \emptyset = \bigcap B \quad \vee \quad \bigcap B \in \mathbb{V},$$

⁵A similar axiomatization, but for positive *comprehension*, is given by M. Forti and R. Hinnion in [FH89]. On the other hand, no finite axiomatization exists for *generalized positive comprehension*, as O. Esser has shown in [Ess04].

because $\forall a \in B \ x \in a$ is clearly a GPF with parameter B , and the intersection is a subclass of any $c \in B$.

To prove the converse, assume now that Θ holds. Since it is not yet clear what we can do with sets, we have to be pedantic with respect to Cartesian products. We define

$$A \times_2 B = \{\langle a, b_1, b_2 \rangle \mid a \in A, \langle b_1, b_2 \rangle \in B\},$$

which is not the same as $A \times B$ for $B \subseteq \mathbb{V}^2$, because $\langle a, b_1, b_2 \rangle = \langle \langle a, b_1 \rangle, b_2 \rangle$, whereas the elements of $A \times B$ are of the form $\langle a, \langle b_1, b_2 \rangle \rangle$. Yet we can construct this and several other set theoretic operations from Θ :

$$\begin{aligned} a \times_2 b &= \{\langle z, x, y \rangle \mid \langle x, y, z \rangle \in b \times a\} \\ a \cup b &= \bigcup \{a, b\} \\ a \cap b &= \bigcup \bigcup \{\{x\} \mid x \in a \cap b\} = \bigcup \bigcup (\Delta \cap (a \times b)) \\ a \cap \mathbb{V}^2 &= a \cap \left(\bigcup \bigcup a \right)^2 \\ \{\{x\} \mid \{x\} \in a\} &= a \cap \bigcup \left(\Delta \cap \left(\bigcup a \right)^2 \right) \\ \text{dom}(a) &= \bigcup \left\{ \{x\} \mid \{x\} \in \bigcup (a \cap \mathbb{V}^2) \right\} \\ a^{-1} &= \text{dom}(\{\langle y, x, z \rangle \mid \langle x, y, z \rangle \in a \times \{a\}\}) \end{aligned}$$

We will prove by induction that for all GPF $\phi(X_1, \dots, X_m, Y_1, \dots, Y_n)$ with parameters Y_1, \dots, Y_n and free variables X_1, \dots, X_m , and for all classes B_1, \dots, B_n and sets a_1, \dots, a_m ,

$$A_{a_1, \dots, a_m}^\phi = \{\langle x_1, \dots, x_m \rangle \in a_1 \times \dots \times a_m \mid \phi(x_1, \dots, x_m, B_1, \dots, B_n)\}$$

is T -closed. This will prove the specification axiom for ϕ , because

$$\{x \in c \mid \phi(x, b_2, \dots, b_m, B_1, \dots, B_n)\} = \text{dom} \left(\dots \text{dom} \left(A_{c, \{b_2\}, \dots, \{b_m\}}^\phi \right) \dots \right),$$

where the domain operation is applied $m - 1$ times.

Each induction step will reduce the claim to a subformula or to a formula with fewer quantifiers. Let us assume wlog that no bound variable is among the X_1, \dots or Y_1, \dots and just always denote the bound variable in question by Z .

Case 1: Assume ϕ is $\forall Z \in Y_i \ \psi$. Then

$$A_{a_1, \dots, a_m}^\phi = \bigcap_{x \in B_i} \text{dom} \left(A_{a_1, \dots, a_m, \{x\}}^{\psi(Z/X_{m+1})} \right),$$

where $\psi(Z/X_{m+1})$ is the formula ψ , with each free occurrence of Z substituted by X_{m+1} . This is the step which is only needed for GPF formulas. Since it is the only point in the proof where we make use of the closure axiom, we otherwise still obtain BPF specification as claimed in the theorem.

Case 2: Assume ϕ is a bounded quantification. If ϕ is $\exists Z \in X_i \ \psi$, then

$$A_{a_1, \dots, a_m}^\phi = \text{dom} \left(A_{a_1, \dots, a_m, b}^{\psi(Z/X_{m+1}) \wedge X_{m+1} \in X_i} \right),$$

where $b = \bigcup a_i$. If ϕ is $\forall Z \in X_i \ \psi$, then

$$A_{a_1, \dots, a_m}^\phi = \text{dom} \left\{ \langle x, y \rangle \in a_1 \times \dots \times a_m \times a_i \mid \forall z \in y \ \langle x, y, z \rangle \in A_{a_1, \dots, a_m, a_i, b}^\rho \right\},$$

where again $b = \bigcup a_i$, and ρ is the formula $\psi(Z/X_{m+2}) \wedge X_{m+1} = X_i$. The class defined here is of the form $\{\langle x, y \rangle \in b \mid \forall z \in y \ \langle x, y, z \rangle \in a\}$ and therefore a set, by our assumption.

Case 3: Assume ϕ is a conjunction or disjunction. If ϕ is $\psi \wedge \chi$ resp. $\psi \vee \chi$, then

$$A_{a_1, \dots, a_m}^\phi = A_{a_1, \dots, a_m}^\psi \cap A_{a_1, \dots, a_m}^\chi \quad \text{resp.} \quad A_{a_1, \dots, a_m}^\phi = A_{a_1, \dots, a_m}^\psi \cup A_{a_1, \dots, a_m}^\chi.$$

Case 4: Assume ϕ is atomic. If X_m does not occur in ϕ , then $A_{a_1, \dots, a_m}^\phi = A_{a_1, \dots, a_{m-1}}^\phi \times a_m$. If ϕ has more than one variable, but X_{m-1} is not among them, then:

$$A_{a_1, \dots, a_m}^\phi = \left\{ \langle z, x_{m-1}, x_m \rangle \mid \langle z, x_m, x_{m-1} \rangle \in A_{a_1, \dots, a_{m-2}, a_m}^{\phi(X_m/X_{m-1})} \times a_{m-1} \right\}$$

Applying these two facts recursively reduces the problem to the case where either $m = 1$ or where X_m and X_{m-1} both occur in ϕ :

$$\begin{aligned} A_{a_1}^{X_1=X_1} &= a_1 \\ A_{a_1}^{X_1 \in X_1} &= \text{dom}(E \cap a_1^2) \\ A_{a_1, \dots, a_m}^{X_{m-1}=X_m} &= a_1 \times \dots \times a_{m-2} \times_2 (\Delta \cap (a_{m-1} \times a_m)) \\ A_{a_1, \dots, a_m}^{X_m=X_{m-1}} &= a_1 \times \dots \times a_{m-2} \times_2 (\Delta \cap (a_{m-1} \times a_m)) \\ A_{a_1, \dots, a_m}^{X_{m-1} \in X_m} &= a_1 \times \dots \times a_{m-2} \times_2 (E \cap (a_{m-1} \times a_m)) \\ A_{a_1, \dots, a_m}^{X_m \in X_{m-1}} &= a_1 \times \dots \times a_{m-2} \times_2 (E^{-1} \cap (a_{m-1} \times a_m)) \end{aligned}$$

□

As we already indicated, the theory GPK^+ uses GPF *comprehension*, but if $\mathbb{V} \in \mathbb{V}$, specification entails comprehension. GPK^+ can be axiomatized as follows:

- $\mathbb{V} \in \mathbb{V}$
- $\mathbb{A} = \emptyset \in \mathbb{V}$
- GPF specification

Proposition 14. GPK^+ implies TS and that unions of sets are sets.

Proof. If $B \subseteq \mathbb{T}$, then $\bigcap B = \{x \mid \forall y \in B \ x \in y\}$ is \mathbb{T} -closed, and if $a, b \in \mathbb{T}$, then $a \cup b = \{x \mid x \in a \vee x \in b\} \in \mathbb{V}$, because these are defined by GPFs, proving the 2nd and 3rd topology axioms. $\{a\} = \{x \mid x=a\}$ and $x=a$ is bounded positive, so T_1 is also true.

$\square a \cap \diamond b = \{c \mid \exists x \in b \ x=c \wedge \forall x \in c \ x \in a \wedge \exists x \in b \ x \in c\}$ is defined by a positive formula as well, yielding the exponential axiom.

$\bigcup a = \{c \mid \exists x \in a \ c \in x\}$ is also \mathbb{T} -closed, for the same reason.

The formula $z = \{x, y\}$ can be expressed as $x \in z \wedge y \in z \wedge \forall w \in z \ (w=x \vee w=y)$, so it is bounded positive. Using that, we see that ordered pairs, Cartesian products, domains and ranges can all be defined by GPFs. This allows us to prove the additivity axiom:

Let $a \in \mathbb{T}$ be discrete and $F : a \rightarrow \mathbb{V}$. We first show that $F \in \mathbb{V}$: Firstly, $F \subseteq a \times \mathbb{V}$ and $a \times \mathbb{V}$ is \mathbb{T} -closed. Secondly, if $\langle x, y \rangle \in (a \times \mathbb{V}) \setminus F$, then $F(x) \neq y$, so F is a subclass of the \mathbb{T} -closed $(a \setminus \{x\} \times \mathbb{V}) \cup \{\langle x, F(x) \rangle\}$, which does not contain $\langle x, y \rangle$. Thus F is a set and hence $\bigcup \text{rng}(F)$ is \mathbb{T} -closed. □

5 Regularity and Union

After having seen that topological set theory is provable in GPK^+ , we now aim for a result in the other direction. To this end we assume in addition to ES the union axiom and that every set is a regular space:

$$\begin{aligned} \text{Union} \quad & \bigcup a \text{ is } \mathbb{T}\text{-closed for every } a \in \mathbb{V}. \\ T_3 \quad & x \in a \wedge b \in \square a \Rightarrow \exists u, v. u \cup v = a \wedge x \notin u \wedge b \cap v = \emptyset \end{aligned}$$

These two axioms elegantly connect the topological and set-theoretic properties of orders and products. Note that they, too, are theorems of ZF, because every discrete set is regular and its union is a set.

Recall that we use the term *ordered set* only for sets with an order \leq , whose order-topology is at least as fine as their natural topology. By default, we consider the order itself to be the non-strict version.

Proposition 15 (ES + Union + T₃). 1. Domains and ranges of sets are sets.

2. Every map in \mathbb{V} is continuous and closed with respect to the natural topology.
3. A linear order \leq on a set a is a set iff its order topology is at most as fine as the natural topology of a .
4. The product topology of a^n is equal to the natural topology.
5. If \mathbb{A} is closed, GPF specification holds.

Proof. (1): Let a be a set. Then $c = \bigcup \bigcup a$ is a set, and in fact, $c = \text{dom}(a) \cup \text{rng}(a)$. But $\text{dom}(a) = \bigcup (\square_{\leq 1} c \cap \bigcup a)$, which proves that domains of sets are sets. Now $F_{2,2,1} \upharpoonright c^2 : c^2 \rightarrow c^2$ is a set, and so is $(c^2 \times a) \cap F_{2,2,1}$. But the domain of this set is a^{-1} , and the domain of a^{-1} is $\text{rng}(a)$.

(2): Let $f \in \mathbb{V}$ be a map from a to b , and let $c \subseteq b$ be closed. Then $f \cap (a \times c)$ is a set, too, and so is $f^{-1}[c] = \text{dom}(f \cap (a \times c))$. Thus f is continuous. Similarly, if $c \subseteq a$ is closed, then $f[c] = \text{rng}(f \cap (c \times b))$ is a set and hence f is closed.

(3): Now let a be linearly ordered by \leq . If $x \in a$, then $[x, \infty) = \text{rng}(\{x\} \times a \cap \leq)$ and $(-\infty, x] = \text{dom}((a \times \{x\}) \cap \leq)$. Conversely assume that all intervals $[x, y]$ are sets. Then if $\langle x, y \rangle \in a^2 \setminus \leq$, that is, $x > y$. If there is a $z \in (y, x)$, then $(z, \infty) \times (-\infty, z)$ is a relatively open neighborhood of $\langle x, y \rangle$ disjoint from \leq . Otherwise, $(y, \infty) \times (-\infty, x)$ is one.

(4): To show that the topologies on a^n coincide, we only need to consider the case $n = 2$; the rest follows by induction, because products of regular spaces are regular. Since a is Hausdorff, we already know from Proposition 1 that the universal topology is at least as fine as the product topology, and it remains to prove the converse.

Let $b \subseteq a^2$ be a set. We will show that it is closed with respect to the product topology. Let $\langle x, y \rangle \in a^2 \setminus b$. Then $x \notin \text{dom}(b \cap (a \times \{y\}))$, so by regularity, there is a closed neighborhood $u \ni x$ disjoint from that set. Thus $b \cap (a \times \{y\}) \cap (u \times a) = \emptyset$, that is, $y \notin \text{rng}(b \cap (u \times a))$. Again by T₃, there is a closed neighborhood $v \ni y$ disjoint from that. Hence $b \cap (u \times v) = \emptyset$ and $u \times v$ is a neighborhood of $\langle x, y \rangle$ with respect to the product topology.

(5): We only have to prove Θ from Theorem 13: The statements about the permutations of triples are true because the topologies on products coincide. $\Delta \cap a$ is closed in $(\text{dom}(a) \cup \text{rng}(a))^2$, even with respect to the product topology, because every set is Hausdorff. $E \cap a$ is a set by regularity: If $\langle x, y \rangle \in a \setminus E$, then $x \neq y$, so x and y can be separated by disjoint $U \ni x$ and $V \ni y$ relatively open in $\text{dom}(a) \cup \text{rng}(a)$. $a \cap (U \times V)$ is a neighborhood of $\langle x, y \rangle$ disjoint from E .

It remains to show that $B = \{\langle x, y \rangle \in b \mid \forall z \in y \langle x, y, z \rangle \in a\}$ is closed for every $a \in \mathbb{V}$. Since

$$B = b \cap \{\langle x, y \rangle \in c^2 \mid \forall z \in y \langle x, y, z \rangle \in a \cap c^3\},$$

where $c = \text{dom}(b) \cup \text{rng}(b) \cup \bigcup \text{rng}(b)$, we can wlog assume that $b = c^2$ and $a \subseteq c^3$, and prove that B is a closed subset of c^2 . Let $\langle x, y \rangle \in c^2 \setminus B$, that is, let $\exists z \in y \langle x, y, z \rangle \notin a$. By (4) there exist relatively open neighborhoods U, V and W of x, y and z in c , such that $U \times V \times W$ is disjoint from a . But then $c \cap \Diamond W$ equals $c \setminus (A \cup \square(c \setminus W))$ or $c \setminus (A \cup \{\emptyset\} \cup \square(c \setminus W))$, depending on whether $\emptyset \in \mathbb{V}$, so $c \cap \Diamond W$ is relatively open and hence $U \times (V \cap \Diamond W)$ is an open neighborhood of $\langle x, y \rangle$ in c^2 disjoint from B . \square

Together with (5), Proposition 14 thus proves:

Corollary 16. $\text{GPK}_{(\infty)}^+ + T_3$ is equivalent to $\text{TS}_{(\infty)} + (\mathbb{A} = \emptyset \in \mathbb{V}) + \text{Union} + T_3$.

6 Uniformization

Choice principles in the presence of a universal set are problematic. By Theorem 6, for example, $\mathbb{V} \in \mathbb{V}$ implies that there is a perfect set and in particular that not every set is well-orderable. And in [FH96a, FH98, Ess00], M. Forti, F. Honsell and O. Esser identified plenty of choice principles as inconsistent with positive set theory. On the other hand, many topological arguments rely on

some kind of choice. The following *uniformization axiom* turns out to be consistent and yet have plenty of convenient topological implications, in particular with regard to compactness.

A *uniformization* of a relation $R \subseteq \mathbb{V}^2$ is a function $F \subseteq R$ with $\text{dom}(F) = \text{dom}(R)$. The *uniformization axiom* states that we can simultaneously choose elements from a family of classes as long as it is indexed by a discrete set:

Uniformization If $\text{dom}(R)$ is a discrete set, R has a uniformization.

Unless the relation is empty, its uniformization will be a set by the additivity axiom. Therefore the uniformization axiom can be expressed with at most one universal and no existential quantification over classes, and thus still be equivalently formulated in a first-order way, using axiom schemes. Let us denote by ESU resp. TSU essential resp. topological set theory with uniformization.

In these theories, at least all discrete sets are well-orderable. The following proof goes back to S. Fujii and T. Nogura ([FN99]). We call $f : \square a \rightarrow a$ a *choice function* if $f(b) \in b$ for every $b \in \square a$.

Proposition 17 (ESU). *A set a is well-orderable iff it is Hausdorff and there exists a continuous choice function $f : \square a \rightarrow a$, such that $b \setminus \{f(b)\}$ is closed for all b .*

In particular, every discrete set is well-orderable and in bijection to κ^\oplus for some cardinal κ .

Proof. If a is well-ordered, we only have to define $F(b) = \min(b)$. In a well-order, the minimal element is always isolated, so $b \setminus F(b)$ is in fact closed. To show that F is a set, let $c \subseteq a$ be closed. Then the preimage of c consists of all nonempty subsets of a whose minimal element is in c . Assume $b \notin F^{-1}[c]$, that is $F(b) \notin c$. Then

$$(\square a \cap \diamond((-\infty, F(b)] \cap c)) \cup \square[F(b) + 1, \infty)$$

is a closed superset of $F^{-1}[c]$ omitting b , where by $F(b) + 1$ we denote the successor of $F(b)$, and if $F(b)$ is the maximal element, we consider the right part of the union to be empty. Hence $F^{-1}[c]$ is in fact closed, proving that F is continuous and a set.

For the converse, assume now that f is a continuous choice function. A set $p \subseteq \square a$ is an *approximation* if:

- $a \in p$
- p is well-ordered by reverse inclusion \supseteq .
- For every nonempty proper initial segment $Q \subset p$, we have $\bigcap Q \in p$.
- For every non-maximal $b \in p$, we have $b \setminus \{f(b)\} \in p$.

We show that two approximations p and q are always initial segments of one another, so they are well-ordered by inclusion: Let Q be the initial segment they have in common. Since both contain $b = \bigcap Q$, that intersection must be in Q and hence the maximal element of Q . If b is not the maximum of either p or q , both contain $b \setminus \{f(b)\}$, which is a contradiction because that is not in Q .

Thus the union P of all approximations is well-ordered. Assume $\bigcap P$ has more than one element. Then $P \cup \{\bigcap P, \bigcap P \setminus f(\bigcap P)\}$ were an approximation strictly larger than P . Thus $\bigcap P$ is empty or a singleton. Since there is no infinite descending chain, and for every bounded ascending chain $Q \subseteq P$, we have $\bigcap Q \in P$, P is closed, so $P \in \mathbb{V}$. Also, \supseteq is a set-well-order on P . Thus a is also set-well-orderable, because $f \upharpoonright P$ is a continuous bijection onto a :

Firstly, it is injective, because after the first b with $f(b) = x$, x is omitted. Secondly, it is surjective, because if $b \in P$ is the first element not containing x , it cannot be the intersection of its predecessors and thus has to be of the form $b = c \setminus \{f(c)\}$. Hence $x = f(c)$. If x is a member of every element of P , then $\bigcap P = \{x\} \in P$ and $x = f(\{x\})$.

Now let a be a discrete set. We only have to prove that a continuous choice function $f : \square a \rightarrow a$ exists. In fact, any choice function will do, since $\square a$ is discrete and hence every function on $\square a$ is continuous. And the existence of such a function follows from the uniformization axiom, applied to the relation $R \subseteq \square d \times d$ defined by: xRy iff $y \in x$.

It follows that every discrete set a is well-orderable. Therefore, it is comparable in length to On . If an initial segment of a were in bijection to On , then as the image of a discrete set, On would be a set. Hence a must be in bijection to a proper initial segment α^\oplus of On . If κ is the cardinality of α , there is a bijection between κ^\oplus and α^\oplus . Composing these bijections proves the claim. \square

It follows that there exists an infinite discrete set iff $\omega \in On$. The uniformization axiom also allows us to define for every infinite cardinal κ a cardinal 2^κ , namely the least ordinal in bijection to $\square\kappa^\oplus$. Proposition 17 then shows that, just as in ZFC, On is not only a weak but even a strong limit.

Like the axiom of choice, the uniformization axiom could be stated in terms of products. Of course, it only speaks of products of \mathcal{D} -few factors at first, but surprisingly it even has implications for larger products as long as the factors are indexed by a \mathcal{D} -compact well-ordered set. \mathcal{D} -compactness for a well-ordered set just means that no subclass of cofinality $\geq On$ is closed.

Proposition 18 (ESU + T_3 + Union). *Let w be a \mathcal{D} -compact well-ordered set, $a \in \mathbb{V}$ and $a_x \subseteq a$ a nonempty for every $x \in w_I$. Then the product $\prod_{x \in w_I} a_x$ is nonempty.*

Proof. Recall that the product is defined as:

$$\prod_{x \in w_I} a_x = \{F \cup (w' \times a) \mid F : w_I \rightarrow \mathbb{V}, \forall x F(x) \in a_x\}$$

We do induction on the length of w and we have to distinguish three cases:

Case 1: If w has no greatest element, its cofinality must be \mathcal{D} -small or else it would not be \mathcal{D} -compact Hausdorff. So let $\langle y_\alpha \mid \alpha < \kappa \rangle$ be a cofinal strictly increasing sequence. Using the induction hypothesis and the uniformization axiom, choose for every $\alpha < \kappa$ an element

$$f_\alpha \in \prod_{x \in]y_\alpha, y_{\alpha+1}]_I} a_x$$

Then the union of the f_α is an element of $\prod_{x \in w_I} a_x$.

Case 2: Assume that w has a greatest element p and that $w \setminus \{p\}$ is a set. Then this is still \mathcal{D} -compact Hausdorff and hence the induction hypothesis applies, so there is an element $f : w \setminus \{p\} \rightarrow a$ of the product missing the last dimension. For any $y \in a_p$, the set $f \cup \langle p, y \rangle$ is in $\prod_{x \in w_I} a_x$.

Case 3: Finally assume that w has a greatest element p and that $w \setminus \{p\}$ is not a set. By the induction hypothesis,

$$P_y = \prod_{x \in [-\infty, y]_I} a_x$$

is a nonempty set for every $y < p$. The union $Q = \bigcup_{y < p} P_y$ is not a set, because otherwise its domain $\text{dom}(\bigcup Q) = w \setminus \{p\}$ would also be a set. But since $Q \subseteq \square(w \times a)$, it does have a closure which is a set, and this closure must have an element g with $p \in \text{dom}(g)$. We will show that $f = g \cup (w' \times a)$ witnesses the claim, that is, $f \in \prod_{x \in w_I} a_x$.

If $z \in w_I$, then g is not in the closure of $\bigcup_{y < z} P_y$, because that is a subclass of the set $\square((-\infty, z] \times a)$. Thus g is in the closure of $\bigcup_{z \leq y < p} P_y$, which is a subclass of:

$$M_z = \square(w \times a) \cap \{r \mid r \cap (\{z\} \times a) \in \square_{\leq 1} a_z\}$$

If we can show that M_z is closed, we can deduce that $g \in M_z$ for every $z \in w_I$ and therefore $g \upharpoonright w_I = f \upharpoonright w_I$ is a function from w_I to a with $f(x) \in a_x$ for all $x \in w_I$. Thus f is indeed an element of the product.

To prove that M_z is closed in $\square(w \times a) \cap \diamond(\{z\} \times a_z)$, assume r is an element of the latter but not of the former. Then there are distinct $x_1, x_2 \in a_z$, such that $\langle z, x_1 \rangle, \langle z, x_2 \rangle \in r$. Since a_z is Hausdorff, there are u_1 and u_2 , such that $x_1 \notin u_1, x_2 \notin u_2$ and $u_1 \cup u_2 = a$, and

$$\square((w \setminus \{z\}) \times a \cup \{z\} \times u_1) \cup \square((w \setminus \{z\}) \times a \cup \{z\} \times u_2)$$

is a closed superset of M_z omitting r . \square

Some of the known models of topological set theory are ultrametrizable, which in the presence of the uniformization axiom is a very strong topological property. A set a is *ultrametrizable* if there is a decreasing sequence $\langle \sim_\alpha \mid \alpha \in On \rangle$ of equivalence relations on a such that $\bigcap_\alpha \sim_\alpha = \Delta_a$ and the α -balls $[x]_\alpha = \{y \mid x \sim_\alpha y\}$ for $x \in a$ and $\alpha \in On$ are a base of the natural topology on a in the sense of open classes, that is, the relatively open classes $U \subseteq a$ are exactly the unions of balls. If that is the case, the α -balls partition a into clopen sets for every α .

Proposition 19 (ESU). *Every ultrametrizable set is a \mathcal{D} -compact linearly orderable set.*

Proof. For every $\alpha \in On$, the class C_α of all α -balls is a subclass of $\Box a$. If $b \in \Box a$ and $x \in b$, then $\Diamond[x]_\alpha$ is a neighborhood of b in $\Box a$ which contains only one element of C_α , namely $[x]_\alpha$. Hence C_α has no accumulation points and is therefore a discrete set. That means there are only \mathcal{D} -few α -balls for every $\alpha \in On$.

Now let $A \subseteq \Box a$ and $\bigcap A = \emptyset$. For each α , let B_α be the union of all α -balls which intersect every element of A . Then $\bigcap_\alpha B_\alpha = \emptyset$ and every B_α is closed.

Assume that all B_α are nonempty. Then for every α all but \mathcal{D} -few members of the sequence $\langle B_\alpha \mid \alpha \in On \rangle$ are elements of the closed set $\Box B_\alpha$, so every accumulation point must be in $\bigcap_{\alpha \in On} \Box B_\alpha$, which is empty. Thus $\{B_\alpha \mid \alpha \in On\}$ has no accumulation point and is a discrete subset of $\Box B_0$. Hence it is \mathcal{D} -small, which means that the sequence $\langle B_\alpha \mid \alpha \in On \rangle$ is eventually constant, a contradiction.

Therefore there is a B_α which is empty, and by definition every α -ball is disjoint from some element of A . Since there are only \mathcal{D} -few α -balls, the uniformization axiom allows us to choose for every α -ball $[x]_\alpha$ an element $c_{[x]_\alpha} \in A$ disjoint from $[x]_\alpha$. The set of these $c_{[x]_\alpha}$ is discrete and has an empty intersection. This concludes the proof of the \mathcal{D} -compactness.

Since it is discrete, the set C_α can be linearly ordered and there are only \mathcal{D} -few such linear orders for every α . If L is a linear order on C_α , let R_L be the partial order relation on a defined by $x R_L y$ iff $[x]_\alpha L [y]_\alpha$. R_L is a set because it is the union of \mathcal{D} -few sets of the form $[x]_\alpha \times [y]_\alpha$. Let S_α be the set of all such R_L . The sequence $\langle S_\alpha \mid \alpha \in On \rangle$ can only be eventually constant if a is discrete, in which case it is linearly orderable anyway. If a is not discrete, however, $S = \bigcup_\alpha S_\alpha$ must be \mathcal{D} -large and therefore have an accumulation point \leq in $\Box a^2$. Because each S_α is \mathcal{D} -small, \leq is in the closure of every $\bigcup_{\beta > \alpha} S_\beta$. For $x, y \in a$, let

$$t_{\alpha, x, y} = \Box(a^2 \setminus ([y]_\alpha \times [x]_\alpha)) \cap \Diamond\{\langle x, y \rangle\}.$$

We will show that \leq is a linear order on a :

Assume $x \neq y$. Then there is an α such that $x \sim_\alpha y$. Every element of $\bigcup_{\beta > \alpha} S_\beta$ assigns an order to $[x]_\alpha$ and $[y]_\alpha$, so it is in exactly one of the disjoint closed sets $t_{\alpha, x, y}$ and $t_{\alpha, y, x}$. Therefore the same must be true of \leq , so we have $x \leq y$ iff not $y \leq x$. This proves antisymmetry and totality.

If $x \leq y \leq z$ and x, y, z are distinct, then there is an α such that $x \sim_\alpha y \sim_\alpha z \sim_\alpha x$. Then \leq is in the closure of neither $t_{\alpha, y, x}$ nor $t_{\alpha, z, y}$, and must therefore be in the closure of

$$\bigcup_{\beta > \alpha} S_\beta \cap t_{\alpha, x, y} \cap t_{\alpha, y, z},$$

which is a subset of $t_{\alpha, x, z}$, because every element of S is transitive. It follows that \leq is also in $t_{\alpha, x, z}$ and thus $x \leq z$, proving transitivity.

Finally, \leq is reflexive because for every $x \in a$, all of S lies in the set $\Box a^2 \cap \Diamond\{\langle x, x \rangle\}$. \square

Another consequence of the uniformization axiom is the following law of distributivity:

Lemma 20 (ESU). *If d is discrete and for each $i \in d$, J_i is a nonempty class, then*

$$\bigcup_{i \in d} \bigcap_{j \in J_i} j = \bigcap_{f \in \prod_{i \in d} J_i} \bigcup_{i \in I} f(i).$$

Proof. If x is in the set on the left, there exists an $i \in d$ such that x is an element of every $j \in J_i$. Thus for every function f in the product, $x \in f(i)$. Hence x is an element of the right hand side.

Conversely, assume that x is not in the set on the left, that is, for every $i \in d$, there is a $j \in J_i$ such that $x \notin j$. Let f be a uniformization of the relation $R = \{\langle i, j \rangle \mid i \in d, x \notin j \in J_i\}$. Then $x \notin \bigcup_{i \in I} f(i)$. \square

It implies that we can work with subbases in the familiar way. Let us call \mathcal{K} *regular* if every union of \mathcal{K} -few \mathcal{K} -small sets is \mathcal{K} -small again. Then in particular \mathcal{D} is regular.

Lemma 21 (ESU). *Let $\mathcal{K} \subseteq \mathcal{D}$ and let B be a \mathcal{K} -subbase of a topology T such that the union of \mathcal{K} -few elements of B always is an intersection of elements of B . Then B is a base of T .*

Proof. We only have to prove that the intersections of elements of B are closed with respect to \mathcal{K} -small unions and therefore constitute a \mathcal{K} -topology. But if I is \mathcal{K} -small, and each $\langle b_{i,j} \mid j \in J_i \rangle$ is a family in B , we have

$$\bigcup_{i \in I} \bigcap_{j \in J_i} b_{i,j} = \bigcap_{f \in \prod_{i \in I} J_i} \bigcup_{i \in I} b_{i,f(i)}$$

by Lemma 20, and every \mathcal{K} -small union $\bigcup_{i \in I} b_{i,f(i)}$ is an element of B again. \square

Thus if \mathcal{K} is regular and S is a \mathcal{K} -subbase of T , the class of all \mathcal{K} -small unions of elements of S is a base of T . Since $\bigcup_i \Diamond a_i = \Diamond \bigcup_i a_i$, the sets of the following form constitute a base of the exponential \mathcal{K} -topology:

$$\Diamond_T a \cup \bigcup_{i \in I} \Box_T b_i,$$

where I is \mathcal{K} -small and $a, b_i \in T$ for all $i \in I$. As that is sometimes more intuitive, we also use open classes in our arguments instead of closed sets. By setting $U = \mathcal{C}a$ and $V_i = \mathcal{C}b_i$, we obtain that every open class is a union of classes of the following form:

$$\Box_T U \cap \bigcap_{i \in I} \Diamond_T V_i$$

That is, these constitute a base in the sense of *open classes*. Since $\Box U = \Box U \cap \Diamond U$, the class U can always be assumed to be the union of the V_i .

Lemma 21 also implies that given a class B , the weak comprehension principle suffices to prove the existence of the topology \mathcal{K} -generated by B : A set c is closed iff for every $x \in \mathcal{C}a$, there is a discrete family $(b_i)_{i \in I}$ in B , such that $c \subseteq \bigcup_i b_i$ and $x \notin \bigcup_i b_i$. In particular, the \mathcal{K} -topology of $\text{Exp}_{\mathcal{K}}(X)$ exists (as a class) whenever the topology of X is a set.

Lemma 22 (ESU). *Let \mathcal{K} be regular and X a \mathcal{K} -topological T_0 -space.*

1. *If X is T_1 , then $\text{Exp}_{\mathcal{K}}(X)$ is T_1 (but not necessarily conversely).*
2. *X is T_3 iff $\text{Exp}_{\mathcal{K}}(X)$ is T_2 .*
3. *X is T_4 iff $\text{Exp}_{\mathcal{K}}(X)$ is T_3 .*

Proof. In this proof we use \Box and \Diamond with respect to X , not the universe, so if T is the topology of X , we set $\Box a = \Box_T a$ and $\Diamond a = \Diamond_T a$.

(1): For $a \in \text{Exp}_{\mathcal{K}}(X)$, the singleton $\{a\} = \Box a \cap \bigcap_{x \in a} \Diamond \{x\}$ is closed in $\text{Exp}_{\mathcal{K}}(X)$.

(As a counterexample to the converse consider the case where $\mathcal{K} = \kappa$ is a regular cardinal number and $X = (\kappa + 1)^\oplus$, with the κ -topology generated by the singletons $\{\alpha\}$ for $\alpha < \kappa$. This is not T_1 , because $\{\kappa\}$ is not closed, but it is clearly T_0 . We show that its exponential κ -topology is T_1 : Let $a \in \text{Exp}_{\mathcal{K}}(X)$. Then either $a \subseteq \kappa$ is small or $a = X$.

In the first case, $\{a\} = \Box a \cap \bigcap_{x \in a} \Diamond \{x\}$ is closed. In the second case, $\{a\} = \{X\} = \bigcap_{x \in \kappa} \Diamond \{x\}$ is also closed.)

(2): (\Rightarrow) Let $a, b \in \text{Exp}_{\mathcal{K}}(X)$ be distinct, wlog $x \in b \setminus a$. Then there are disjoint open $U, V \subseteq X$ separating x from a . Hence $\Diamond U$ and $\Box V$ separate b from a .

(\Leftarrow) Firstly, we have to show that X is T_1 . Assume that $\{y\}$ is not closed, so there exists some other $x \in \text{cl}(\{y\})$, and by T_0 , y is not in the closure of x , so $\text{cl}(\{x\}) \subset \text{cl}(\{y\})$. The two closures can

be separated by open base classes $\Box U \cap \bigcap_i \Diamond U_i$ and $\Box V \cap \bigcap_j \Diamond V_j$ of $\text{Exp}_{\mathcal{K}}(X)$, whose intersection $\Box(U \cap V) \cap \bigcap_i \Diamond U_i \cap \bigcap_j \Diamond V_j$ is empty. Hence there either exists a U_i disjoint from V – which is impossible because $\text{cl}(\{x\}) \in \Box V \cap \bigcap_i \Diamond U_i \rightarrow$, or there is a V_j disjoint from U : But since $V_j \cap \text{cl}(\{y\}) \neq \emptyset$, we have $y \in V_j$. Hence $y \notin U \ni x$, contradicting the assumption that x is in the closure of y .

Now let $x \notin a$. Then a and $b = \{x\} \cup a$ can be separated by open base classes $\Box U \cap \bigcap_i \Diamond U_i$ and $\Box V \cap \bigcap_j \Diamond V_j$ of $\text{Exp}_{\mathcal{K}}(X)$, whose intersection $\Box(U \cap V) \cap \bigcap_i \Diamond U_i \cap \bigcap_j \Diamond V_j$ is empty. Hence there either exists a U_i disjoint from V – which is impossible because $a \in \Box V \cap \bigcap_i \Diamond U_i \rightarrow$, or there is a V_j disjoint from U : Then V_j and U separate x from a , because b meets V_j and a does not, so $x \in V_j$.

(3): In both directions, the T_1 property follows from the previous points.

(\Rightarrow) Let $a \notin c$, $a \subseteq X$ closed and $c \subseteq \text{Exp}_{\mathcal{K}}(X)$ closed. Wlog⁶ let c be of the form $\Box b$ or $\Diamond b$ with closed $b \subseteq X$. In the first case, $a \not\subseteq b$, so let U, V separate some $x \in a \setminus b$ from b . Then $\Diamond U, \Box V$ separate $\{a\}, c$. In the second case, $a \cap b = \emptyset$, so let U, V separate them. Then $\Box U, \Diamond V$ separate $\{a\}, c$.

(\Leftarrow) Now let $\text{Exp}_{\mathcal{K}}(X)$ be T_3 and let $a, b \subseteq X$ be closed, nonempty and disjoint. Then $\{a\}$ and $\Diamond b$ are disjoint and can be separated by disjoint open $U, V \subseteq \text{Exp}_{\mathcal{K}}(X)$. U can be assumed to be an open base class, so $U = \Box W \cap \bigcap_i \Diamond W_i$. We claim that $\text{cl}(W) \cap b = \emptyset$, which proves the normality of X . So assume that there exists $x \in \text{cl}(W) \cap b$. Then $a \cup \{x\} \in \Diamond b$, so one of the open base classes $\Box Z \cap \bigcap_j \Diamond Z_j$ constituting V must contain $a \cup \{x\}$. That means that either one of the Z_j must be disjoint from W – which is impossible because $x \in Z_j$ – or one of the W_i must be disjoint from Z – which also cannot be the case, because all W_i intersect a and $a \subseteq Z$. \square

7 Compactness

Hyperuniverses are \mathcal{D} -compact Hausdorff spaces, so \mathcal{D} -compactness is another natural axiom to consider. In the case $\mathbb{V} \notin \mathbb{V}$, the corresponding statement would be that every set is \mathcal{D} -compact (note that this is another axiom provable in ZFC), but if $\mathbb{V} \in \mathbb{V}$, this is equivalent to \mathbb{V} being \mathcal{D} -compact. And in fact, TSU with a \mathcal{D} -compact Hausdorff \mathbb{V} implies most of the additional axioms we have looked at so far, including the separation properties and the union axiom:

Let $a \subseteq \mathbb{T}$ and $x \notin \bigcup a$. Then for every $y \in a$, there is a b such that $y \subseteq \text{int}(b)$ and $x \notin b$. The sets $\Box \text{int}(b)$ then cover a and by \mathcal{D} -compact Hausdorffness, a discrete subfamily also does. But then the union of these b is a superset of $\bigcup a$ not containing x .

Another consequence of global \mathcal{D} -compactness is that most naturally occurring topologies coincide: Point (2) of the following theorem not only applies to hyperspaces $\Box a$, but also to products, order topologies and others. If the class of atoms is closed and unions of sets are sets, this even characterizes compactness (note that these two assumptions are only used in (3) \Rightarrow (1)):

Theorem 23 (ESU + T_2 + Union). *If \mathbb{A} is \mathbb{T} -closed, the following statements are equivalent:*

1. *Every set is \mathcal{D} -compact, that is: If $\bigcap A = \emptyset$, there is a discrete $d \subseteq A$ with $\bigcap d = \emptyset$.*
2. *Every Hausdorff \mathcal{D} -topology $T \in \mathbb{V}$ equals the natural topology: $T = \Box \bigcup T$*
3. *For every set a , the exponential \mathcal{D} -topology on $\Box a$ equals the natural topology.*

Proof. (1) \Rightarrow (2): Let $A = \bigcup T$. Since A is T -closed in A , $A \in T$ and thus $A \in \mathbb{V}$. By definition, $T \subseteq \Box A$. For the converse, we have to verify that each $b \in \Box A$ is T -closed, so let $y \in A \setminus b$. Consider the class C of all $u \in T$, such that there is a $v \in T$ with $u \cup v = A$ and $y \notin v$. By the Hausdorff axiom, for every $x \in b$ there is a $u \in C$ omitting x , so $b \cap \bigcap C = \emptyset$. By \mathcal{D} -compactness, there is a discrete $d \subseteq C$ with $b \cap \bigcap d = \emptyset$. By definition of C , $y \in \text{int}_T(u)$ for every u , and

⁶To verify that a space X is T_3 it suffices to separate each point x from each subbase set b not containing x : Firstly, the \mathcal{K} -small unions of subbase sets b are a base, so if x is not in a \mathcal{K} -small union $\bigcup_i b_i$, it can be separated with U_i, V_i from every b_i , and $\bigcap U_i, \bigcup V_i$ separate x from the union. This shows that x can then be separated from each base set. Secondly, every closed set is an intersection $\bigcap_i b_i$ of base sets b_i , and if x is not in that intersection, there is an i with $x \notin b_i$ and if U_i, V_i separate x from b_i , they also separate x from $\bigcap_i b_i$.

since d is discrete, the intersection $\bigcap_{u \in d} \text{int}_T(u)$ is open. Therefore, every $y \notin b$ has a T -open neighborhood disjoint from b .

(2) \Rightarrow (3) is trivial, because as a \mathcal{D} -compact Hausdorff \mathcal{D} -topological space, a is T_3 and hence $\Box a$ is Hausdorff by Lemma 22.

(3) \Rightarrow (1): Lemma 22 also implies that if $\Box \Box a$ is T_2 , then $\Box a$ is T_3 and a is T_4 , so it follows from the Hausdorff axiom that every set is normal.

Finally, we can prove \mathcal{D} -compactness. Let $A \subseteq \Box a$, $\bigcap A = \emptyset$ and let $c = \text{cl}(A)$. Then $\bigcap c = \emptyset$. Since every set is regular and \mathbb{A} is closed, the positive specification principle holds. Therefore

$$B = \left\{ b \in \Box c \mid \bigcap b \neq \emptyset \right\} = \{ b \in \Box c \mid \exists x \forall y \in b \ x \in y \}$$

is a closed subset of $\Box c$ not containing c . In particular, there is an open base class

$$\Box U \cap \bigcap_{i \in I} \Diamond V_i$$

of the space $\Box c$ containing c which is disjoint from B . Every $U \cap V_i$ is a relatively open subset of c , so there is an $x_i \in A \cap U \cap V_i$, because A is dense in c . The set $\{x_i \mid i \in I\}$ – and here we used the uniformization axiom – then is a discrete subcover of A . \square

References

- [Ess97] O. Esser. An interpretation of the Zermelo-Fraenkel set theory and the Kelley-Morse set theory in a positive theory. *MLQ: Mathematical Logic Quarterly*, 43:369–377, 1997.
- [Ess99] O. Esser. On the consistency of a positive theory. *MLQ: Mathematical Logic Quarterly*, 45:105–116, 1999.
- [Ess00] O. Esser. Inconsistency of the axiom of choice with the positive theory GPK_{∞}^+ . *The Journal of Symbolic Logic*, 65(4):1911–1916, 2000.
- [Ess03] O. Esser. Mildly ineffable cardinals and hyperuniverses. *Reports on Mathematical Logic*, 37:23–39, 2003.
- [Ess04] O. Esser. *Une théorie positive des ensembles*. Cahiers du Centre de logique. Academia-Bruylant, 2004.
- [Fac12] A. Fackler. *Topological set theories and hyperuniverses*. PhD thesis, Ludwig-Maximilians-Universität München, 2012. <http://edoc.ub.uni-muenchen.de/14258>.
- [FH89] M. Forti and R. Hinnion. The consistency problem for positive comprehension principles. *The Journal of Symbolic Logic*, 54(4):1401–1418, 1989.
- [FH96a] M. Forti and F. Honsell. Choice principles in hyperuniverses. *Annals of Pure and Applied Logic*, 77(1):35–52, 1996.
- [FH96b] M. Forti and F. Honsell. A general construction of hyperuniverses. *Theor. Comput. Sci.*, 156:203–215, March 1996.
- [FH98] M. Forti and F. Honsell. Addendum and corrigendum: Choice principles in hyperuniverses. *Annals of Pure and Applied Logic*, 92(2):211–214, 1998.
- [FHL96] M. Forti, F. Honsell, and M. Lenisa. Axiomatic characterizations of hyperuniverses and applications. In *University of Southern*, pages 140–163. Society Press, 1996.
- [FN99] S. Fujii and T. Nogura. Characterizations of compact ordinal spaces via continuous selections. *Topology and its Applications*, 91(1):65 – 69, 1999.
- [Mal76] R. J. Malitz. *Set theory in which the axiom of foundation fails*. PhD thesis, UCLA, 1976.
- [Wey89] E. Weydert. *How to Approximate the Naive Comprehension Scheme inside of Classical Logic*. PhD thesis, Friedrich-Wilhelms-Universität Bonn, 1989.